Math 310
9-30-2020

Review Sheet for Test 2

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. Premises: $\begin{cases} (\neg P \vee Q) \to (R \wedge \neg S) \\ \neg S \to \neg R \end{cases}$

Prove: P.

2. Premises: $\begin{cases} (\neg C \vee \neg D) \to B \\ B \to A \\ \neg A \vee C \end{cases}$

Prove: C.

3. Premises: $\begin{cases} (\neg P \vee S) \to R \\ Q \to \neg P \\ Q \vee R \end{cases}$

Prove: R.

4. Let x and y be positive real numbers. Prove that if $\dfrac{1}{x} + \dfrac{1}{y} = 2$, then either $x \geq 1$ or $y \geq 1$.

5. Prove that the following inequality has no real solutions:

$$x^2y^2 + 6 + x^2 - 4xy + y^2 < 0.$$

6. **Rolle's theorem** says that if f is a continuous function on the interval $a \leq x \leq b$, f is differentiable on the interval $a < x < b$, and $f(a) = f(b)$, then $f'(c) = 0$ for some number c such that $a < c < b$.

Use Rolle's theorem to prove that if $k > 0$, then $f(x) = x^5 + x^3 + kx + 1$ does not have more than one root.

7. Prove that if an integer is squared and divided by 3, the remainder can't be 2. [Hint: Take cases. If the given integer is divided by 3, it leaves a remainder of 0, 1, or 2.]

8. Prove that if n is an integer, then $n^2 + 2n + 2$ is not divisible by 3.

9. Prove that if $x \in \mathbb{R}$, then

$$-4 \leq |x - 2| - |x - 6| \leq 4.$$

10. Prove that for all $x \in \mathbb{R}$, $|x + 1| + |2x - 6| \geq 4$.

11. Prove that if $n \in \mathbb{Z}$ and $n \geq 1$, then $x^n - y^n$ is divisible by $x - y$. Note: This means that there's a polynomial $p(x, y)$ in x and y with integer coefficients such that $(x - y) \cdot p(x, y) = x^n - y^n$.

12. Prove that $\displaystyle\sum_{k=1}^{n} (-1)^{k+1} k^2 = (-1)^{n+1} \frac{n(n+1)}{2}$ for all $n \geq 1$.

13. Prove that if $n \in \mathbb{N}$, then

$$\sum_{k=1}^{n} (k^2 + k + 1)k! = (n + 1)(n + 1)! - 1.$$

14. A sequence of integers is defined by

$$x_0 = 7, \quad x_1 = -10,$$

$$x_n = -4x_{n-1} + 12x_{n-2} \quad \text{for} \quad n \geq 2.$$

Prove that for all $n \geq 0$,

$$x_n = 3 \cdot (-6)^n + 4 \cdot 2^n.$$

15. Prove that if $n \geq 2$, then

$$5^n > 2^n + 3^n.$$

16. (a) What would be a counterexample to the statement "Every dog likes cheese"?

(b) Give a counterexample to the following statement: "For all integers a, b, and c, if a divides bc, then a divides b or a divides c."

17. Give a specific counterexample which shows that the following equations are *not* algebraic identities.

(a) "$2(xy) = (2x)(2y)$".

(b) "$\sin(x + y) = \sin x + \sin y$".

18. Give counterexamples to the following statements:

(a) "If $x, y \in \mathbb{R}$ and $x \leq y$, then $\sin x \leq \sin y$."

(b) "If $a, b, c, d \in \mathbb{R}$ and $a \leq b$ and $c \leq d$, then $ac \leq bd$."

19. Give a counterexample to the following statement: "If $x^2 - 4x + 3 = 0$, then $x = 1$."

20. Suppose the universe is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and:

$A = \{1, 2, 3, 4, 5\}$,

$B = \{2, 4, 6, 8, 10\}$,

$C = \{1, 3, 5, 7, 9\}$.

(a) List the elements of $\overline{A}$.

(b) List the elements of $A \cap C$.

(c) List the elements of $A \cup B$.

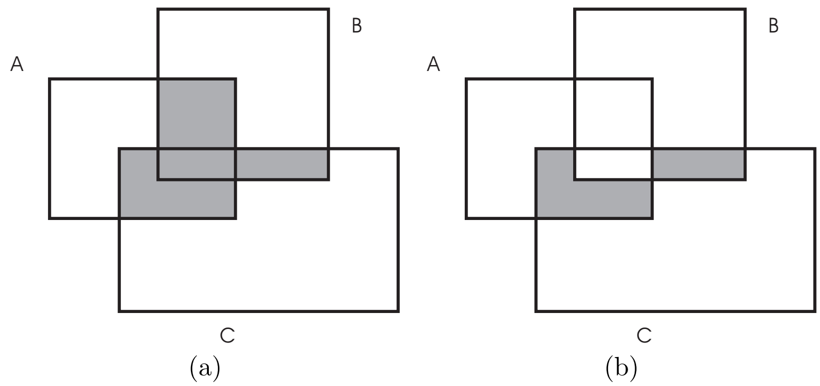
(d) List the elements of $(A \cup C) - B$.

21. Construct Venn diagrams for the following sets:

(a) $A - (B - C)$

(b) $A \cap (B \cup C)$

22. What sets are represented by the shaded regions in the following Venn diagrams?



23. (a) List the elements of the set $\{a, \{b, c\}\}$.

(b) List the elements of the set $\{\{d\}\}$.

(c) Is $\{a, b\}$ a subset of $\{a, \{b, c\}\}$? Why or why not?

24. (a) Suppose $A = \{1, a, \emptyset\}$. List the elements of $\mathcal{P}(A)$.

(b) How many subsets does the set $\{n \in \mathbb{N} \mid 2 \leq n \leq 10\}$ have?

25. (a) Suppose $X = \{a, b\}$ and $Y = \{1, 2, 42\}$. List the elements of $X \times Y$.

(b) Are the sets $\{1, 2\}$ and $\{2, 1\}$ equal? Are the ordered pairs $(1, 2)$ and $(2, 1)$ equal?

(c) Suppose $|X| = 5$ and $|Y| = 4$. Find $|X \times Y|$ and $|\mathcal{P}(X \times Y)|$.

26. Let A, B, and C be sets. Prove that

$$A \cap B \subset (A \cup C) \cap (B \cup C).$$

27. Let A, B, and C be sets. Prove that

$$(C - A) \cup B = (A \cap B) \cup [(C \cup B) - A].$$

28. Let A and B be sets. Prove that

$$(A - B) \cap (B - A) = \emptyset.$$

29. Let A and B be sets.

(a) Prove that $A \subset A \cup B$ and $B \subset A \cup B$.

(b) Prove that $A \cup B = B$ if and only if $A \subset B$.

30. Recall that

$$(a, b) = \{x \in \mathbb{R} \mid x > a \wedge x < b\}.$$

Prove that $(1, 3) \cap (2, 4) = (2, 3)$.

31. Recall that

$$[a, b] = \{x \in \mathbb{R} \mid x \geq a \wedge x \leq b\}.$$

Prove that $[-2, 3] \cup [1, 5] = [-2, 5]$.

32. Suppose $S = \{1, 2\}$ and $T = \{a, b\}$. List the elements of $S \times T$ and $T \times S$.

33. Give a specific example of two sets A and B for which $A \times B \neq B \times A$.

34. Prove using the limit definition that

$$\lim_{n \to \infty} \frac{3n + 5}{n + 1} = 3.$$

35. Prove using the limit definition that

$$\lim_{n \to \infty} \frac{10n}{2n + 1} = 5.$$

Solutions to the Review Sheet for Test 2

1. Premises: $\begin{cases} (\neg P \lor Q) \to (R \land \neg S) \\ \neg S \to \neg R \end{cases}$

Prove: P.

1.	$\neg P \lor Q) \to (R \land \neg S)$	Premise
2.	$\neg S \to \neg R$	Premise
3.	$\neg P$	Premise for proof by contradiction
4.	$\neg P \lor Q$	Constructing a disjunction (3)
5.	$R \land \neg S$	Modus ponens (1,4)
6.	R	Decomposing a conjunction (5)
7.	$\neg S$	Decomposing a conjunction (5)
8.	S	Modus tollens (2,6)
9.	$S \land \neg S$	Constructing a conjunction (7,8)
10.	P	Proof by contradiction (3,9) $\square$

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2. Premises: $\begin{cases} (\neg C \lor \neg D) \to B \\ B \to A \\ \neg A \lor C \end{cases}$

Prove: C.

1.	$B \to A$	Premise
2.	$\neg A \lor C$	Premise
3.	$(\neg C \lor \neg D) \to B$	Premise
4.	B	Premise for proof by cases
5.	A	Modus ponens (1, 4)
6.	C	Disjunctive syllogism (2, 5)
7.	$\neg B$	Premise for proof by cases
8.	$\neg(\neg C \lor \neg D)$	Modus tollens (3, 7)
9.	$C \land D$	DeMorgan's law (8)
10.	C	Decomposing a conjunction (9)
11.	C	Proof by cases (4, 6, 7, 10) $\square$

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3. Premises: $\begin{cases} (\neg P \lor S) \to R \\ Q \to \neg P \\ Q \lor R \end{cases}$

Prove: R.

1.	$(\neg P \vee S) \to R$	Premise
2.	$Q \to \neg P$	Premise
3.	$Q \vee R$	Premise
4.	P	Premise for proof by cases
5.	$\neg Q$	Modus tollens (2, 4)
6.	R	Disjunctive syllogism (3, 5)
7.	$\neg P$	Premise for proof by cases
8.	$\neg P \vee S$	Constructing a disjunction (7)
9.	R	Modus ponens (1, 8)
10.	R	Proof by cases (4, 6, 7, 9) $\square$

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4. Let x and y be positive real numbers. Prove that if $\dfrac{1}{x} + \dfrac{1}{y} = 2$, then either $x \ge 1$ or $y \ge 1$.

Suppose that x and y are positive real numbers, and $\dfrac{1}{x} + \dfrac{1}{y} = 2$. I want to prove that either $x \ge 1$ or $y \ge 1$.

I will give a proof by contradiction. The negation of "Either $x \ge 1$ or $y \ge 1$" is "$x < 1$ and $y < 1$". So suppose on the contrary that $x < 1$ and $y < 1$.

Since $x < 1$ and x is positive, $\dfrac{1}{x} > 1$. Since $y < 1$ and y is positive, $\dfrac{1}{y} > 1$. Therefore,

$$\frac{1}{x} + \frac{1}{y} > 1 + 1 = 2.$$

This contradicts my assumption that $\dfrac{1}{x} + \dfrac{1}{y} = 2$. Therefore, either $x \ge 1$ or $y \ge 1$. $\square$

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5. Prove that the following inequality has no real solutions:

$$x^2y^2 + 6 + x^2 - 4xy + y^2 < 0.$$

I'll use proof by contradiction. Suppose that (x, y) is a solution to the inequality. I'll rewrite the inequality using basic algebra, the idea being to try to *complete the squares*:

$$\begin{aligned} x^2y^2 + 6 + x^2 - 4xy + y^2 &< 0 \\ x^2y^2 - 2xy + 6 + x^2 - 2xy + y^2 &< 0 \\ x^2y^2 - 2xy + 6 + (x - y)^2 &< 0 \\ x^2y^2 - 2xy + 1 + (x - y)^2 + 5 &< 0 \\ (xy - 1)^2 + (x - y)^2 + 5 &< 0 \end{aligned}$$

The last inequality gives a contradiction: The two square terms must be greater than or equal to 0, so adding 5 makes the left side strictly positive. However, the inequality says that the left side is negative.

This contradiction shows that the original inequality has no solutions. $\square$

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6. **Rolle's theorem** says that if f is a continuous function on the interval $a \le x \le b$, f is differentiable on the interval $a < x < b$, and $f(a) = f(b)$, then $f'(c) = 0$ for some number c such that $a < c < b$.

Use Rolle's theorem to prove that if $k > 0$, then $f(x) = x^5 + x^3 + kx + 1$ does not have more than one root.

Suppose on the contrary that f has more than one root. Then f must have *at least two different roots*, say a and b. Thus, $f(a) = 0$ and $f(b) = 0$. I'll assume that $a < b$; if it's the other way around, just switch their names.

Now f is a polynomial, so it's differentiable and continuous everywhere. In addition, $f(a) = 0 = f(b)$. Therefore, Rolle's theorem applies, and I know that $f'(c) = 0$ for some c such that $a < c < b$.

On the other hand,

$$f'(x) = 5x^4 + 3x^2 + k > 0 \quad \text{for all} \quad x.$$

Reason: The first two term are positive numbers multiplied by even powers of x, so they're both greater than or equal to 0. The last term k was given to be greater than 0.

Therefore, $f'(x)$ can't be equal to 0, which contradicts the existence of a number c where $f'(c) = 0$.

Therefore, f does not have more than one root. $\square$

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7. Prove that if an integer is squared and divided by 3, the remainder can't be 2.

Let n be an integer. I want to show that n^2 does not leave a remainder of 2 when it's divided by 3.

When n is divided by 3, it can leave a remainder of 0, 1, or 2. I consider cases.

If n leaves a remainder of 0 when it's divided by 3, then $n = 3k$ for some integer k. Hence,

$$n^2 = (3k)^2 = 9k^2 = 3(3k^2).$$

Therefore, n^2 leaves a remainder of 0 when it's divided by 3.

If n leaves a remainder of 1 when it's divided by 3, then $n = 3k + 1$ for some integer k. Hence,

$$n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1.$$

Therefore, n^2 leaves a remainder of 1 when it's divided by 3.

Finally, if n leaves a remainder of 2 when it's divided by 3, then $n = 3k + 2$ for some integer k. Hence,

$$n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 9k^2 + 12k + 3 + 1 = 3(3k^2 + 4k + 1) + 1.$$

Therefore, n^2 leaves a remainder of 1 when it's divided by 3.

I've exhausted all the cases. Hence, if n is an integer, then n^2 does not leave a remainder of 2 when it's divided by 3. $\square$

Note: This kind of problem is much easier to do using **modular arithmetic**.

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8. Prove that if n is an integer, then $n^2 + 2n + 2$ is not divisible by 3.

Every integer n can be written in one of the forms $3q$, $3q + 1$, or $3q + 2$, where q is an integer.

Case 1. If $n = 3q$, then

$$n^2 + 2n + 2 = 9q^2 + 6q + 2 = 3(3q^2 + 2q) + 2.$$

In this case, $n^2 + 2n + 2$ leaves a remainder of 2 when it's divided by 3.

Case 2. If $n = 3q + 1$, then

$$n^2 + 2n + 2 = 9q^2 + 12q + 5 = 3(3q^2 + 4q + 1) + 2.$$

In this case, $n^2 + 2n + 2$ leaves a remainder of 2 when it's divided by 3.

Case 3. If $n = 3q + 2$, then

$$n^2 + 2n + 2 = 9q^2 + 18q + 10 = 3(3q^2 + 6q + 3) + 1.$$

In this case, $n^2 + 2n + 2$ leaves a remainder of 1 when it's divided by 3.

Therefore, if n is an integer, $n^2 + 2n + 2$ is not divisible by 3. $\square$

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9. Prove that if $x \in \mathbb{R}$, then

$$-4 \le |x - 2| - |x - 6| \le 4.$$

Case 1. $x < 2$.

In this case

$$|x - 2| = -(x - 2) = -x + 2 \quad \text{and} \quad |x - 6| = -(x - 6) = -x + 6.$$

So

$$|x - 2| - |x - 6| = (-x + 2) - (-x + 6) = -4.$$

Therefore, $-4 \le |x - 2| - |x - 6| \le 4$.

Case 2. $2 \le x \le 6$.

In this case

$$|x - 2| = x - 2 \quad \text{and} \quad |x - 6| = -(x - 6) = -x + 6.$$

So

$$|x - 2| - |x - 6| = (x - 2) - (-x + 6) = 2x - 8.$$

Now

$$\begin{aligned} 2 &\le x \le 6 \\ 4 &\le 2x \le 12 \\ -4 &\le 2x - 8 \le 4 \end{aligned}$$

Therefore, $-4 \le |x - 2| - |x - 6| \le 4$.

Case 3. $x > 6$.

In this case

$$|x - 2| = x - 2 \quad \text{and} \quad |x - 6| = x - 6.$$

So

$$|x - 2| - |x - 6| = (x - 2) - (x - 6) = 4.$$

Therefore, $-4 \le |x - 2| - |x - 6| \le 4$.

Since the result holds in all three cases, and the cases cover all $x \in \mathbb{R}$, it follows that $-4 \le |x-2|-|x-6| \le 4$ for all $x \in \mathbb{R}$. $\square$

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10. Prove that for all $x \in \mathbb{R}$, $|x + 1| + |2x - 6| \ge 4$.

(It is *not* a proof to draw the graph!)

I'll take cases to get rid of the absolute values.

$$|x + 1| = \begin{cases} x + 1 & \text{if } x \ge -1 \\ -(x + 1) & \text{if } x < -1 \end{cases} \quad \text{and} \quad |2x - 6| = \begin{cases} 2x - 6 & \text{if } x \ge 3 \\ -(2x - 6) & \text{if } x < 3 \end{cases}.$$

My cases will be $x < -1$, $-1 \le x < 3$, and $x \ge 3$. This accounts for all $x \in \mathbb{R}$.

Suppose that $x < -1$. Then $|x+1| = -(x+1)$, and since $x < -1 < 3$, $|2x-6| = -(2x-6)$. Therefore,

$$|x+1| + |2x-6| = -(x+1) - (2x-6) = 5 - 3x.$$

Now

$$\begin{aligned} x &< -1 \\ -3x &> 3 \\ 5 - 3x &> 8 > 4 \\ 5 - 3x &\ge 4 \\ |x+1| + |2x-6| &\ge 4 \end{aligned}$$

Hence, the result is true for $x < -1$.

Suppose that $-1 \le x < 3$. Since $x \ge -1$, $|x+1| = x+1$. Since $x < 3$, $|2x-6| = -(2x-6)$. Therefore,

$$|x+1| + |2x-6| = (x+1) - (2x-6) = 7 - x.$$

Now

$$\begin{aligned} -1 \le x &< 3 \\ 1 \ge -x &> -3 \\ 8 \ge 7 - x &> 4 \\ 7 - x &\ge 4 \\ |x+1| + |2x-6| &\ge 4 \end{aligned}$$

Hence, the result is true for $-1 \le x < 3$.

Finally, suppose that $x \ge 3$. This implies that $|2x-6| = 2x-6$. Moreover, since $x \ge 3 > -1$, $|x+1| = x+1$. Therefore,

$$|x+1| + |2x-6| = (x+1) + (2x-6) = 3x - 5.$$

Now

$$\begin{aligned} x &\ge 3 \\ 3x &\ge 9 \\ 3x - 5 &\ge 4 \\ |x+1| + |2x-6| &\ge 4 \end{aligned}$$

Hence, the result is true for $x \ge 3$.

Since in all three cases I have $|x+1| + |2x-6| \ge 4$, this is true for all $x \in \mathbb{R}$. $\square$

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11. Prove that if $n \in \mathbb{Z}$ and $n \ge 1$, then $x^n - y^n$ is divisible by $x - y$.

I'll use induction on n.

For $n = 1$, $x^n - y^n = x - y$, which is obviously divisible by $x - y$. For $n = 2$,

$$x^n - y^n = x^- y^2 = (x-y)(x+y).$$

This is also divisible by $x - y$.

Let $n > 2$, and assume the result is true for all powers less than n. In particular, assume that $x^{n-2} - y^{n-2}$ and $x^{n-1} - y^{n-1}$ are divisible by $x - y$. I want to prove that $x^n - y^n$ is divisible by $x - y$.

Since $x^{n-2} - y^{n-2}$ and $x^{n-1} - y^{n-1}$ are divisible by $x - y$, there are polynomials $p(x,y)$ and $q(x,y)$ in the variables x and y such that

$$x^{n-2} - y^{n-2} = p(x,y)(x-y),$$

$$x^{n-1} - y^{n-1} = q(x,y)(x-y) \qquad (*).$$

I'll use (*) to build the expression $x^n - y^n$ that I'm interested in. I'll make an x^n first: Multiply (*) by x:

$$x^n - xy^{n-1} = xq(x,y)(x-y), \quad x^n = xy^{n-1} + xq(x,y)(x-y).$$

Next, make the y^n: Multiply (*) by y:

$$x^{n-1}y - y^n = yq(x,y)(x-y), \quad -y^n = -x^{n-1}y + yq(x,y)(x-y).$$

Now add the expressions for x^n and $-y^n$ and do some algebra, substituting eventually for $x^{n-2} - y^{n-2}$:

$$x^n - y^n = xy^{n-1} - x^{n-1}y + xq(x,y)(x-y) + yq(x,y)(x-y) = xy(y^{n-2} - x^{n-2}) + (x+y)q(x,y)(x-y) =$$

$$-xyp(x,y)(x-y) + (x+y)q(x,y)(x-y).$$

Both terms on the right side are divisible by $x-y$, so $x^n - y^n$ is divisible by $x-y$. By induction, $x^n - y^n$ is divisible by $x-y$ for all $n \ge 1$. □

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12. Prove that $\displaystyle\sum_{k=1}^{n}(-1)^{k+1}k^2 = (-1)^{n+1}\frac{n(n+1)}{2}$ for all $n \ge 1$.

I'll use induction on n. For $n = 1$,

$$\sum_{k=1}^{n}(-1)^{k+1}k^2 = (-1)^2 1^2 = 1, \quad \text{while} \quad \frac{n(n+1)}{2} = \frac{1\cdot 2}{2} = 2.$$

Let $n > 1$, and suppose the result is true for $n-1$. Thus, assume that

$$\sum_{k=1}^{n-1}(-1)^{k+1}k^2 = (-1)^n\frac{(n-1)n}{2}.$$

I want to prove that result for n. Start with the summation for n, "peel off" the n^{th} term, and use the induction hypothesis:

$$\sum_{k=1}^{n}(-1)^{k+1}k^2 = \sum_{k=1}^{n-1}(-1)^{k+1}k^2 + (-1)^{n+1}n^2 = (-1)^n\frac{(n-1)n}{2} + (-1)^{n+1}n^2 = (-1)^n\cdot n\cdot\left(\frac{n-1}{2} - n\right) =$$

$$(-1)^n\cdot n\cdot\left(-\frac{1}{2} - \frac{n}{2}\right) = (-1)^n\cdot n\cdot\left(-\frac{n+1}{2}\right) = (-1)^{n+1}\frac{n(n+1)}{2}.$$

This proves the result for n, so the result is true for all $n \ge 1$, by induction. □

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13. Prove that if $n \in \mathbb{N}$, then

$$\sum_{k=1}^{n}(k^2 + k + 1)k! = (n+1)(n+1)! - 1.$$

For $n = 1$, I have

$$\sum_{k=1}^{1}(k^2 + k + 1)k! = 3\cdot 1! = 3 \quad \text{and} \quad (1+1)(1+1)! - 1 = 4 - 1 = 3.$$

Hence, the result is true for $n = 1$.
Assume the result for n:

$$\sum_{k=1}^{n}(k^2 + k + 1)k! = (n + 1)(n + 1)! - 1.$$

I need to prove the result for $n + 1$:

$$\sum_{k=1}^{n+1}(k^2 + k + 1)k! = (n + 2)(n + 2)! - 1.$$

I have

$$\begin{aligned}\sum_{k=1}^{n+1}(k^2 + k + 1)k! &= \sum_{k=1}^{n}(k^2 + k + 1)k! + [(n + 1)^2 + (n + 1) + 1](n + 1)! \\ &= [(n + 1)(n + 1)! - 1] + [(n + 1)^2 + (n + 1) + 1](n + 1)! \\ &= [(n + 1)^2 + (n + 1) + 1 + (n + 1)](n + 1)! - 1 \\ &= [(n^2 + 2n + 1) + (n + 1) + 1 + (n + 1)](n + 1)! - 1 \\ &= (n^2 + 4n + 4)(n + 1)! - 1 \\ &= (n + 2)(n + 2)(n + 1)! - 1 \\ &= (n + 2)(n + 2)! - 1\end{aligned}$$

This proves the result for $n + 1$, so the result is true for all $n \in \mathbb{N}$, by induction. □

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14. A sequence of integers is defined by

$$x_0 = 7, \quad x_1 = -10,$$

$$x_n = -4x_{n-1} + 12x_{n-2} \quad \text{for} \quad n \geq 2.$$

Prove that for all $n \geq 0$,

$$x_n = 3 \cdot (-6)^n + 4 \cdot 2^n.$$

For $n = 0$, the formula gives

$$x_0 = 3 \cdot (-6)^0 + 4 \cdot 2^0 = 3 + 4 = 7.$$

For $n = 1$, the formula gives

$$x_1 = 3 \cdot (-6)^1 + 4 \cdot 2^1 = -10.$$

Therefore, the result is true for $n = 0$ and $n = 1$.
Assume $n \geq 2$, and suppose the result holds for $k < n$. Then

$$x_n = -4x_{n-1} + 12x_{n-2} = -4\left(3 \cdot (-6)^{n-1} + 4 \cdot 2^{n-1}\right) + 12 \cdot \left(3 \cdot (-6)^{n-2} + 4 \cdot 2^{n-2}\right) =$$

$$\left(-12 \cdot (-6)^{n-1} + 36 \cdot (-6)^{n-2}\right) + \left((-16) \cdot 2^{n-1} + 48 \cdot 2^{n-2}\right) =$$

$$3 \cdot (-6)^{n-2}\left(-4 \cdot (-6) + 12\right) + 4 \cdot 2^{n-2}\left(-4 \cdot 2 + 12\right) =$$

$$3 \cdot (-6)^{n-2} \cdot 36 + 4 \cdot 2^{n-2} \cdot 4 = 3 \cdot (-6)^{n-2} \cdot (-6)^2 + 4 \cdot 2^{n-2} \cdot 2^2 = 3 \cdot (-6)^n + 4 \cdot 2^n.$$

This proves the result for n, so the result holds for all $n \geq 0$ by induction. □

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15. Prove that if $n \geq 2$, then

$$5^n > 2^n + 3^n.$$

For $n = 2$,

$$5^n = 5^2 = 25,$$

$$2^n + 3^n = 2^2 + 3^2 = 13.$$

Thus, $5^2 > 2^2 + 3^2$, and the result is true for $n = 2$.
Assume that the result is true for n:

$$5^n > 2^n + 3^n.$$

I'll prove the result for $n + 1$:

$$\begin{aligned} 5^{n+1} &= 5 \cdot 5^n \\ &> 5(2^n + 3^n) \\ &= 5 \cdot 2^n + 5 \cdot 3^n \\ &> 2 \cdot 2^n + 3 \cdot 3^n \\ &= 2^{n+1} + 3^{n+1} \end{aligned}$$

This proves the result for $n + 1$, so the result is true for all $n \geq 2$ by induction. □

---

16. (a) What would be a counterexample to the statement "Every dog likes cheese"?

(b) Give a counterexample to the following statement: "For all integers a, b, and c, if a divides bc, then a divides b or a divides c."

(a) A counterexample would be a dog who does not like cheese. □

(b) Let $a = 12$, $b = 6$, and $c = 4$. Then 12 divides $6 \cdot 4 = 24$, but 12 does not divide 6 and 12 does not divide 4. □

---

17. Give a specific counterexample which shows that the following equations are *not* algebraic identities.

(a) "$2(xy) = (2x)(2y)$".

(b) "$\sin(x + y) = \sin x + \sin y$".

(a) If $x = 1$ and $y = 3$, then

$$2(xy) = 2(1 \cdot 3) = 2 \cdot 3 = 6, \quad \text{but} \quad (2x)(2y) = (2 \cdot 1)(2 \cdot 3) = (2)(6) = 12. \quad \square$$

(b) If $x = \dfrac{\pi}{2}$ and $y = \dfrac{\pi}{2}$, then

$$\sin(x + y) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \sin \pi = 0, \quad \text{but} \quad \sin x + \sin y = \sin \frac{\pi}{2} + \sin \frac{\pi}{2} = 1 + 1 = 2. \quad \square$$

---

18. Give counterexamples to the following statements:

(a) "If $x, y \in \mathbb{R}$ and $x \leq y$, then $\sin x \leq \sin y$."

(b) "If $a, b, c, d \in \mathbb{R}$ and $a \leq b$ and $c \leq d$, then $ac \leq bd$."

(a) $\dfrac{\pi}{2} < \pi$ (so $\dfrac{\pi}{2} \leq \pi$), but $\sin \dfrac{\pi}{2} = 1$ and $\sin \pi = 0$. Therefore, $\sin \dfrac{\pi}{2} \not\leq \sin \pi$. □

(b) $-1 < 0$ (so $-1 \leq 0$) and $-2 < -1$ (so $-2 \leq -1$), but $(-1)(-2) = 2$, while $(0)(-1) = 0$. Therefore, $(-1)(-2) \not\leq (0)(-1)$. $\square$

---

19. Give a counterexample to the following statement: "If $x^2 - 4x + 3 = 0$, then $x = 1$."

A counterexample must make the "if-then" statement **false**. An "if-then" statement is false exactly when the "if" part is true and the "then" part is false.

If $x = 3$, then $x^2 - 4x + 3 = 9 - 12 + 3 = 0$. Thus, the "if" part is true. But since $x = 3 \neq 1$, the "then" part is false. Therefore, $x = 3$ is a counterexample to the original statement. $\square$

---

20. Suppose the universe is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and:

$A = \{1, 2, 3, 4, 5\}$,

$B = \{2, 4, 6, 8, 10\}$,

$C = \{1, 3, 5, 7, 9\}$.

(a) List the elements of $\overline{A}$.

(b) List the elements of $A \cap C$.

(c) List the elements of $A \cup B$.

(d) List the elements of $(A \cup C) - B$.

(a)

$$\overline{A} = \{6, 7, 8, 9, 10\}. \quad \square$$

(b)

$$A \cap C = \{1, 3, 5\}. \quad \square$$

(c)

$$A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}. \quad \square$$

(d)

$$(A \cup C) - B = \{1, 3, 5, 7, 9\}. \quad \square$$

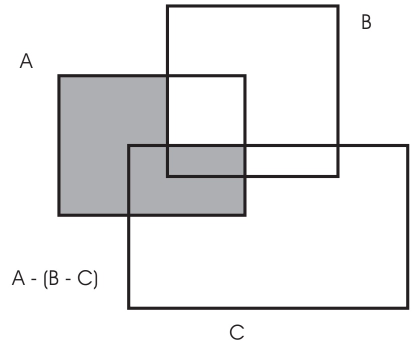
---

21. Construct Venn diagrams for the following sets:

(a) $A - (B - C)$

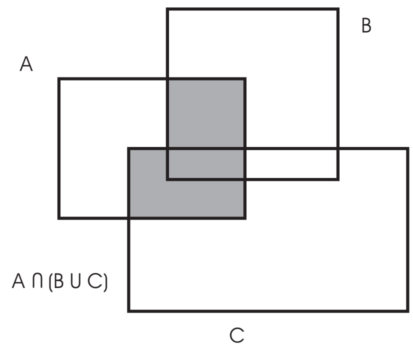
(b) $A \cap (B \cup C)$

(a)



□

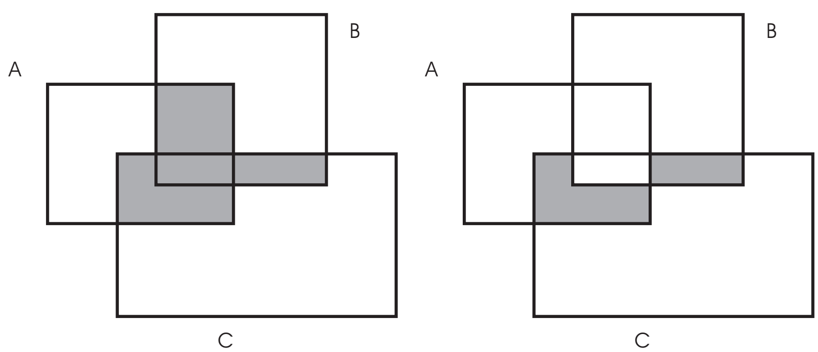
(b)



□

---

22. What sets are represented by the shaded regions in the following Venn diagrams?



(a) (b)

There are many possible answers; here are two.

(a) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$ □

(b) $[(A \cap C) \cup (B \cap C)] - (A \cap B \cap C)$ □

---

23. (a) List the elements of the set $\{a, \{b, c\}\}$.

(b) List the elements of the set $\{\{d\}\}$.

(c) Is $\{a, b\}$ a subset of $\{a, \{b, c\}\}$? Why or why not?

(a) The elements are a and $\{b, c\}$. □

(b) The only element of $\{\{d\}\}$ is $\{d\}$. □

(c) $\{a, b\}$ is not a subset of $\{a, \{b, c\}\}$: b is an element of $\{a, b\}$, but it is not an element of $\{a, \{b, c\}\}$. □

---

24. (a) Suppose $A = \{1, a, \emptyset\}$. List the elements of $\mathcal{P}(A)$.

(b) How many subsets does the set $\{n \in \mathbb{N} \mid 2 \le n \le 10\}$ have?

(a)
$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{a\}, \{\emptyset\}, \{1, a\}, \{1, \emptyset\}, \{a, \emptyset\}, \{1, a, \emptyset\}\}. \quad \square$$

(b) The set has 9 elements (not 8!), so it has $2^9 = 512$ subsets. □

---

25. (a) Suppose $X = \{a, b\}$ and $Y = \{1, 2, 42\}$. List the elements of $X \times Y$.

(b) Are the sets $\{1, 2\}$ and $\{2, 1\}$ equal? Are the ordered pairs$(1, 2)$ and $(2, 1)$ equal?

(c) Suppose $|X| = 5$ and $|Y| = 4$. Find $|X \times Y|$ and $|\mathcal{P}(X \times Y)|$.

(a)
$$X \times Y = \{(a, 1), (a, 2), (a, 42), (b, 1), (b, 2), (b, 42)\}. \quad \square$$

(b) $\{1, 2\} = \{2, 1\}$, because the sets have the same elements. The order in which the elements are listed doesn't matter.

$(1, 2) \neq (2, 1)$ because ordered pairs are equal if and only if their first components are equal and their second components are equal. These two pairs have different first components ($1 \neq 2$) and different second components ($2 \neq 1$). □

(c) First,
$$|X \times Y| = |X| \cdot |Y| = 5 \cdot 4 = 20.$$

Next,
$$|\mathcal{P}(X \times Y)| = 2^{|X \times Y|} = 2^{20} = 1048576. \quad \square$$

---

26. Let A, B, and C be sets. Prove that
$$A \cap B \subset (A \cup C) \cap (B \cup C).$$

Let $x \in A \cap B$. I must show that $x \in (A \cup C) \cap (B \cup C)$.

$$\begin{array}{rccc} x \in A \cap B & \rightarrow & x \in A \wedge x \in B & \text{(Definition of intersection)} \\ & \rightarrow & (x \in A \vee x \in C) \wedge (x \in B \vee x \in C) & \text{(Constructing disjunctions)} \\ & \rightarrow & (x \in A \cup C) \wedge (x \in B \cup C) & \text{(Definition of union)} \\ & \rightarrow & x \in (A \cup C) \cap (B \cup C) & \text{(Definition of intersection)} \end{array}$$

This proves that $A \cap B \subset (A \cup C) \cap (B \cup C)$. □

---

27. Let A, B, and C be sets. Prove that
$$(C - A) \cup B = (A \cap B) \cup [(C \cup B) - A].$$

I'll show each set is contained in the other.

First, I'll show $(C - A) \cup B \subset (A \cap B) \cup [(C \cup B) - A]$.

Let $x \in (C - A) \cup B$. Then $x \in C - A$ or $x \in B$.

Suppose $x \in C - A$. Then $x \in C$, so $x \in C \cup B$ and $x \notin A$, so $x \in (C \cup B) - A$. Hence, $x \in (A \cap B) \cup [(C \cup B) - A]$.

Suppose $x \in B$. If $x \in A$, then $x \in A \cap B$, so $x \in (A \cap B) \cup [(C \cup B) - A]$.

Otherwise, $x \notin A$. Now $x \in B$, so $x \in C \cup B$. Hence, $x \in (C \cup B) - A$. Therefore, $x \in (A \cap B) \cup [(C \cup B) - A]$.

Thus, $x \in (A \cap B) \cup [(C \cup B) - A]$ in every case, so $(C - A) \cup B \subset (A \cap B) \cup [(C \cup B) - A]$.

Conversely, suppose $x \in (A \cap B) \cup [(C \cup B) - A]$. Then $x \in (A \cap B)$ or $x \in (C \cup B) - A$.

Suppose $x \in (A \cap B)$. Then $x \in B$, so $x \in (C - A) \cup B$.

Alternatively, suppose $x \in (C \cup B) - A$. Then $x \in (C \cup B)$ and $x \notin A$. Now $x \in (C \cup B)$ means $x \in C$ or $x \in B$.

In the first case, $x \in C$ and $x \notin A$ give $x \in C - A$, so $x \in (C - A) \cup B$.

In the second case, $x \in B$ gives $x \in (C - A) \cup B$.

In every case, $x \in (C - A) \cup B$, so $(A \cap B) \cup [(C \cup B) - A] \subset (C - A) \cup B$.

Therefore, $(C - A) \cup B = (A \cap B) \cup [(C \cup B) - A]$.

Here is a more formal layout of the proof, with the logical connectives written explicitly.

$$
\begin{aligned}
x \in ((A \cap B) \cup [(C \cup B) - A]) \quad &\leftrightarrow \quad x \in (A \cap B) \vee x \in [(C \cup B) - A] \\
&\qquad \text{Definition of } \cup \\
&\leftrightarrow \quad (x \in A \wedge x \in B) \vee [x \in (C \cup B) \wedge x \notin A] \\
&\qquad \text{Definitions of } \cap \text{, complement} \\
&\leftrightarrow \quad (x \in A \wedge x \in B) \vee [(x \in C \cup x \in B) \wedge x \notin A] \\
&\qquad \text{Definition of } \cup \\
&\leftrightarrow \quad (x \in A \wedge x \in B) \vee [(x \in C \wedge x \notin A) \vee (x \in B \wedge x \notin A)] \\
&\qquad \text{Distributivity} \\
&\leftrightarrow \quad (x \in C \wedge x \notin A) \vee (x \notin A \wedge x \in B) \vee (x \in A \wedge x \in B) \\
&\qquad \text{Commutativity} \\
&\leftrightarrow \quad (x \in C \wedge x \notin A) \vee [(x \notin A \vee x \in A) \wedge x \in B] \\
&\qquad \text{Distributivity} \\
&\leftrightarrow \quad (x \in C \wedge x \notin A) \vee x \in B \\
&\qquad (x \notin A \vee x \in A) \text{ is tautologous} \\
&\leftrightarrow \quad x \in (C - A) \vee x \in B \\
&\qquad \text{Definition of complement} \\
&\leftrightarrow \quad x \in [(C - A) \cup B] \\
&\qquad \text{Definition of } \cup
\end{aligned}
$$

This proves that $(C - A) \cup B = (A \cap B) \cup [(C \cup B) - A]$. $\square$

---

28. Let A and B be sets. Prove that

$$(A - B) \cap (B - A) = \emptyset.$$

Since the empty set is a subset of every set, I know that $\emptyset \subset (A - B) \cap (B - A)$.

Next, I'll show that $(A-B) \cap (B-A) \subset \emptyset$. Taking elements, I have to show that if $x \in (A-B) \cap (B-A)$, then $x \in \emptyset$. (Actually, I'll show that this conditional statement is **vacuously** true by showing that the "if" part is false.)

$$
\begin{array}{rccc}
x \in (A - B) \cap (B - A) & \rightarrow & x \in (A - B) \wedge x \in (B - A) & \text{(Definition of intersection)} \\
& \rightarrow & x \in A \wedge \neg x \in B \wedge x \in B \wedge \neg x \in A & \text{(Definition of complement)} \\
& \rightarrow & (x \in A \wedge \neg x \in A) \wedge (x \in B \wedge \neg x \in B) & \text{(Commutativity)} \\
& \rightarrow & \text{(False)} \wedge \text{(False)} & \text{(Contradiction)} \\
& \rightarrow & \text{(False)} & \text{(Truth table for } \wedge\text{)}
\end{array}
$$

By proof by contradiction, the statement $x \in (A - B) \cap (B - A)$ is false. This makes the conditional statement "if $x \in (A - B) \cap (B - A)$, then $x \in \emptyset$" true! Hence, $(A - B) \cap (B - A) \subset \emptyset$.

Since $(A - B) \cap (B - A) \subset \emptyset$ and $\emptyset \subset (A - B) \cap (B - A)$, it follows that $(A - B) \cap (B - A) = \emptyset$. $\square$

---

29. Let A and B be sets.

(a) Prove that $A \subset A \cup B$ and $B \subset A \cup B$.

(b) Prove that $A \cup B = B$ if and only if $A \subset B$.

(a) Let $x \in A$. Then $x \in A$ or $x \in B$ (the logical rule is "constructing a disjunction"), so $x \in A \cup B$. Therefore, $A \subset A \cup B$.

Let $x \in B$. Then $x \in A$ or $x \in B$ (the logical rule is "constructing a disjunction"), so $x \in A \cup B$. Therefore, $B \subset A \cup B$. $\square$

(b) Suppose $A \cup B = B$. I want to show $A \subset B$.

Let $x \in A$. Then by (a), $x \in A \subset A \cup B = B$. Hence, $A \subset B$.

Suppose $A \subset B$. I want to show $A \cup B = B$.

I will show each of the sets $A \cup B$ and B is contained in the other.

First, by (a) $B \subset A \cup B$.

On the other hand, let $x \in A \cup B$. This means that either $x \in A$ or $x \in B$. In the first case, $x \in A \subset B$. In the second case, $x \in B$. So in either case, $x \in B$. This proves $A \cup B \subset B$.

This proves that $A \cup B = B$. $\square$

---

30. Recall that

$$(a, b) = \{x \in \mathbb{R} \mid x > a \wedge x < b\}.$$

Prove that $(1, 3) \cap (2, 4) = (2, 3)$.

I'll show that $(1, 3) \cap (2, 4) \subset (2, 3)$ and $(2, 3) \subset (1, 3) \cap (2, 4)$.

Suppose $x \in (1, 3) \cap (2, 4)$. Then by the definition of intersection, $x \in (1, 3)$ and $x \in (2, 4)$. By the interval definition, this means that $1 < x$ and $x < 3$ and $2 < x$ and $x < 4$.

In particular, $x < 3$ and $2 < x$, so by the interval definition $x \in (2, 3)$.

This proves that $(1, 3) \cap (2, 4) \subset (2, 3)$.

Conversely, suppose $x \in (2, 3)$, so by the interval definition $2 < x$ and $x < 3$.

First, $x < 3 < 4$. Together with $2 < x$, this means by the interval definition that $x \in (2, 4)$.

Second, $1 < 2 < x$. Together with $x < 3$, this means by the interval definition that $x \in (1, 3)$.

By the definition of intersection, $x \in (1, 3) \cap (2, 4)$.

This proves that $(2, 3) \subset (1, 3) \cap (2, 4)$.

Therefore, $(1, 3) \cap (2, 4) = (2, 3)$. $\square$

---

31. Recall that

$$[a, b] = \{x \in \mathbb{R} \mid x \geq a \wedge x \leq b\}.$$

Prove that $[-2, 3] \cup [1, 5] = [-2, 5]$.

Suppose $x \in [-2, 3] \cup [1, 5]$. Then by the definition of union, $x \in [-2, 3]$ or $x \in [1, 5]$.

If $x \in [-2, 3]$, then by the interval definition $x \geq -2$ and $x \leq 3$. Now $3 < 5$, so $x \leq 5$. Since $x \geq -2$ and $x \leq 5$, by the interval definition $x \in [-2, 5]$.

If $x \in [1, 5]$, then by the interval definition $x \geq 1$ and $x \leq 5$. Now $-2 < 1$, so $x \geq -2$. Since $x \geq -2$ and $x \leq 5$, by the interval definition $x \in [-2, 5]$.

This proves that $[-2, 3] \cup [1, 5] \subset [-2, 5]$.

For the opposite inclusion, suppose $x \in [-2, 5]$. By the interval definition, $x \geq -2$ and $x \leq 5$. Consider two cases.

First, suppose $x \leq 2$. Since $2 < 3$, it follows that $x \leq 3$. Since $x \geq -2$ and $x \leq 3$, by the interval definition $x \in [-2, 3]$. By the definition of union, $x \in [-2, 3] \cup [1, 5]$.

Second, suppose $x > 2$. Now $2 > 1$, so $x \geq 1$. Since $x \geq 1$ and $x \leq 5$, by the interval definition $x \in [1, 5]$. By the definition of union, $x \in [-2, 3] \cup [1, 5]$.

This proves that $[-2, 5] \subset [-2, 3] \cup [1, 5]$.

Hence, $[-2, 3] \cup [1, 5] = [-2, 5]$. □

---

32. Suppose $S = \{1, 2\}$ and $T = \{a, b\}$. List the elements of $S \times T$ and $T \times S$.

$$S \times T = \{(1, a), (1, b), (2, a), (2, b)\} \quad \text{while} \quad T \times S = \{(a, 1), (a, 2), (b, 1), (b, 2)\}. \quad □$$

---

33. Give a specific example of two sets A and B for which $A \times B \neq B \times A$.

For example, let $A = \{a, b\}$ and $B = \{3, 4\}$. Then

$$A \times B = \{(a, 3), (a, 4), (b, 3), (b, 4)\} \quad \text{and} \quad B \times A = \{(3, a), (3, b)(4, a), (4, b)\}.$$

The two sets are not the same. □

---

34. Prove using the limit definition that

$$\lim_{n \to \infty} \frac{3n + 5}{n + 1} = 3.$$

Let $\epsilon > 0$. Set $M = \max\left(0, \frac{2}{\epsilon} - 1\right)$. If $n > M$, then I have

$$n > 0 \quad \text{and} \quad n > \frac{2}{\epsilon} - 1.$$

Hence,

$$\begin{aligned} n &> \frac{2}{\epsilon} - 1 \\ n + 1 &> \frac{2}{\epsilon} \\ \epsilon(n + 1) &> 2 \end{aligned}$$

Since $n > 0$, I have $n + 1 > 0$, so I may divide both sides by $n + 1$, then insert absolute values signs:

$$\begin{aligned} \epsilon &> \frac{2}{n + 1} \\ \epsilon &> \left| \frac{2}{n + 1} \right| \\ \epsilon &> \left| \frac{(3n + 5) - 3(n + 1)}{n + 1} \right| \\ \epsilon &> \left| \frac{3n + 5}{n + 1} - 3 \right| \end{aligned}$$

This proves that $\lim_{n \to \infty} \frac{3n + 5}{n + 1} = 3$. □

35. Prove using the limit definition that

$$\lim_{n\to\infty} \frac{10n}{2n+1} = 5.$$

Let $\epsilon > 0$. Set $M = \max\left(0, \dfrac{5}{2\epsilon} - \dfrac{1}{2}\right)$. If $n > M$, then I have

$$n > 0 \quad \text{and} \quad n > \frac{5}{2\epsilon} - \frac{1}{2}.$$

Hence,

$$\begin{aligned} n &> \frac{5}{2\epsilon} - \frac{1}{2} \\ 2n &> \frac{5}{\epsilon} - 1 \\ 2n+1 &> \frac{5}{\epsilon} \end{aligned}$$

Sine $n > 0$, I have $2n+1 > 0$, so I may multiply both sides by $\dfrac{\epsilon}{2n+1}$ to get

$$\epsilon > \frac{5}{2n+1}.$$

Again, since $2n+1 > 0$, I may insert absolute value signs on the right and obtain

$$\begin{aligned} \epsilon &> \left|\frac{5}{2n+1}\right| \\ \epsilon &> \left|\frac{-5}{2n+1}\right| \\ \epsilon &> \left|\frac{10n - 5(2n+1)}{2n+1}\right| \\ \epsilon &> \left|\frac{10n}{2n+1} - 5\right| \end{aligned}$$

This proves that

$$\lim_{n\to\infty} \frac{10n}{2n+1} = 5. \quad \square$$

He who has overcome his fears will truly be free. - Aristotle

