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Closed Sets and Limit Points

Definition. If X is a topological space, a subset $A \subset X$ is **closed** if $X - A$ is open.

Example. In the standard topology on $\mathbb{R}$, closed intervals are closed and points are closed.

Note, however, than points need not be closed in an arbitrary topological space.

In $\mathbb{R}^n$, the **closed r-ball centered at x**

$$\overline{B(x;r)} = \{y \in \mathbb{R}^n \mid |x - y| \le r\}$$

is closed. $\square$

Closed sets satisfy complementary properties relative to open sets.

Proposition. Let X be a topological space.

1. X and $\emptyset$ are closed.
2. Finite unions of closed sets are closed.
3. Arbitrary intersections of closed sets are closed.

Proof. 1. X is closed, since $X - X = \emptyset$ is open. $\emptyset$ is closed, since $X - \emptyset = X$ is open.

2. Let $C_1, \ldots, C_n$ be closed. For each i, write $C_i = X - U_i$, where U_i is open. Then

$$X - \bigcup_{i=1}^{n} C_i = X - \bigcup_{i=1}^{n} (X - U_i) = \bigcap_{i=1}^{n} (X - (X - U_i)) = \bigcap_{i=1}^{n} U_i.$$

$\bigcap_{i=1}^n U_i$ is open, because it's a finite intersection of open sets. Therefore, $\bigcup_{i=1}^n C_i$ is closed.

3. Let $\{C_a\}_{a \in A}$ be a family of closed sets. For each $a \in A$, write $C_a = X - U_a$, where U_a is open. Then

$$X - \bigcap_{a \in A} C_a = X - \bigcap_{a \in A} (X - U_a) = X - \left(X - \bigcup_{a \in A} U_a\right) = \bigcup_{a \in A} U_a.$$

$\bigcup_{a \in A} U_a$ is open, because it's a union of open sets. Therefore, $\bigcap_{a \in A} C_a$ is closed. $\square$

It's possible to construct a topology by specifying the closed sets rather than the open sets. Then you can define the open sets to be the complements of closed sets.

Example. Give $\mathbb{R}$ the standard topology. The intervals $\left(-\dfrac{1}{n}, \dfrac{1}{n}\right)$ for $n = 1, 2, \ldots$ are open sets. However, their intersection is

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

which is a closed subset of $\mathbb{R}$.

Likewise, the intervals $\left[\frac{1}{n}, \frac{n-1}{n}\right]$ are closed sets. However, their union is

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \frac{n-1}{n}\right] = (0,1),$$

which is open in $\mathbb{R}$. $\square$

Open sets in the subspace topology are intersections of the subspace with open sets in the big space. The next result says that the same is true of closed sets.

Proposition. Let X be a topological space, let Y be a subspace of X, and let $C \subset Y$. C is closed in Y if and only if $C = Y \cap D$, where D is closed in X.

Note that "closed in Y" means "closed in the *subspace topology* on Y".

Proof. Let D be closed in X. Write $D = X - U$, where U is open in X. Then

$$D \cap Y = (X - U) \cap Y = Y - (Y \cap U).$$

(If $x \in (X - U) \cap Y$, then $x \in Y$ and $x \notin U$, so $x \notin Y \cap U$. This means $x \in Y - (Y \cap U)$. Conversely, if $x \in Y - (Y \cap U)$, then $x \in Y$, and $x \notin U$ since $x \notin Y \cap U$. Therefore, $x \in X - U$, so $x \in (X - U) \cap Y$.)

$Y \cap U$ is open in Y, so $Y - (Y \cap U)$ is closed in Y.

Conversely, suppose C is closed in Y. Then $Y - C$ is open in Y, so $Y - C = Y \cap U$, where U is open in X. Hence,

$$C = Y - (Y - C) = Y - (Y \cap U) = (X - U) \cap Y.$$

$X - U$ is closed in X, so C is the intersection of a closed set in X with Y. $\square$

Proposition. Let X be a topological space, let Y be a subspace of X and let $Z \subset Y$.

1. If Z is open in Y and Y is open in X, then Z is open in X.

2. If Z is closed in Y and Y is closed in X, then Z is closed in X.

Proof. I'll prove the first assertion; the proof of the second is similar.

Suppose Z is open in Y and Y is open in X. $Z = Y \cap U$, where U is open in X. But Y is open in X, so Z is an intersection of two open sets of X — hence, it is open in X. $\square$

Definition. Let X be a topological space, and let $Y \subset X$.

1. The **closure** of Y (denoted $\overline{Y}$ or $\text{cl}(Y)$) is the intersection of all closed sets containing Y.

2. The **interior** of Y (denoted $\dot{Y}$ or $\text{int}(Y)$) is the union of all open sets contained in Y.

Notice that the closed set X contains every subset Y, so the intersection formed in taking the closure is not taken over an empty collection.

Remarks.

1. The closure of a set is closed, and the interior of a set is open.

2. The closure of a set is the smallest closed set containing the set; likewise, the interior of a set is the largest open set contained in the set.

3. A set Y is closed if and only if $Y = \overline{Y}$; a set Y is open if and only if $Y = \dot{Y}$.

Example. In the standard topology on $\mathbb{R}$, the closure of (a, b) is $[a, b]$ and the interior of $[a, b]$ is (a, b).

More interestingly, $\overline{\mathbb{Q}} = \mathbb{R}$. (A set whose closure is the whole space is said to be **dense**; thus, the rationals are dense in the reals.) □

Example. The closure of even a single point can be quite large. For example, give $\mathbb{Z}$ the topology whose open sets are the open intervals $(-n, n)$ for $n \in \mathbb{Z}^+$, together with $\emptyset$ and $\mathbb{Z}$.

What is the closure of $\{1\}$? The only closed sets which contain 1 are $\mathbb{Z}$ and $\mathbb{Z} - (-1, 1)$. Now $(-1, 1) = \{0, \}$, so the intersection of these two sets is $\mathbb{Z} - \{0\}$. Thus, the closure of a single point turns out to be everything in $\mathbb{Z}$ but 0! □

Lemma. Let X be a topological space, let Y be a subspace, and let $Z \subset Y$. Then

$$\text{cl}_Y Z = Y \cap \overline{Z}.$$

Here $\text{cl}_Y Z$ is the closure of Z in Y (with the subspace topology), whereas $\overline{Z}$ is the closure of Z in X.

Proof. $\text{cl}_Y Z$ is closed in Y, so $\text{cl}_Y Z = C \cap Y$, where C is closed in X. Now

$$Z \subset \text{cl}_Y Z = C \cap Y \subset C,$$

so C is a closed set in X containing Z.

Therefore, $\overline{Z} \subset C$, so

$$\overline{Z} \cap Y \subset C \cap Y = \text{cl}_Y Z.$$

Conversely, $\overline{Z}$ is closed in X and contains Z, so $\overline{Z} \cap Y$ is closed set in Y containing Z. Therefore, $\text{cl}_Y Z \subset \overline{Z} \cap Y$.

Hence, $\text{cl}_Y Z = Y \cap \overline{Z}$. □

Terminology.

1. If X is a topological space and $x \in X$, a **neighborhood** of x is any open set containing x.

   Some authors use "neighborhood" to mean a set containing x in its interior.

2. If X and Y are sets, I'll say that X **intersects** Y or X **meets** Y if $X \cap Y \neq \emptyset$.

Lemma. Let X be a topological space, and let $Y \subset X$. $x \in \overline{Y}$ if and only if every neighborhood of x intersects Y.

Proof. Suppose $x \in \overline{Y}$. Let U be a neighborhood of x. Suppose on the contrary that $U \cap Y = \emptyset$. Then $U \subset X - Y$, so $X - U \supset X - (X - Y) = Y$.

Now $X - U$ is a closed set containing Y, so $X - U \supset \overline{Y}$. But then $x \in \overline{Y} \subset X - U$ implies $x \notin U$, contrary to the assumption that U was a neighborhood of x.

Conversely, suppose every neighborhood of x intersects y. Let C be a closed set containing Y. I must show that $x \in C$; since C is arbitrary, this will prove that $x \in \overline{Y}$.

Suppose on the contrary that $x \notin C$. $Y \subset C$ implies $X - Y \supset X - C$ — that is, $X - C$ does not intersect Y. Further, $x \in X - C$, and $X - C$ is open. Thus, $X - C$ is a neighborhood of x which does not meet Y, contrary to assumption. Therefore, $x \in C$, which is what I wanted to show. □

Remark. If $\mathcal{B}$ is a basis for the topology on X, then $x \in \overline{Y}$ if and only if every $B \in \mathcal{B}$ intersects Y.

In a sense, the closure of a set consists of the original set together with other points that are "close to" the original set — you might picture these additional points as "boundary points". I want to make the notion of a "boundary point" precise.

Definition. Let X be a topological space, $Y \subset X$. A point $x \in X$ is a **limit point** of Y is every neighborhood of x intersects Y *in some point other than* x.

Example. In the standard topology on $\mathbb{R}$, every point of $[a, b]$ is a limit point of (a, b).

On the other hand, 0 is not a limit point of $(1, 2)$. For example, the open set $(-0.5, 0.5)$ contains 0, but does not intersect $(1, 2)$.

The set $\left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}^+ \right\}$ has 0 as its only limit point.

$\mathbb{Z}$ has no limit points in $\mathbb{R}$. About every point in $\mathbb{R} - \mathbb{Z}$, you can construct an open interval which does not contain any integers. And if $n \in \mathbb{Z}$, the open interval $(n - 0.5, n + 0.5)$ intersects $\mathbb{Z}$ in the point n.

Every real number is a limit point of $\mathbb{Q}$, since every open interval about a real number contains rational numbers. $\square$

Definition. If Y is a subset of a topological space X, the **derived set** Y' is the set of limit points of Y.

Proposition. If Y is a subset of a topological space X, then

$$\overline{Y} = A \cup A'.$$

Proof. Suppose $x \in Y \cup Y'$.

If $x \in Y$, then $x \in \overline{Y}$.

If $x \notin Y$, then $x \in Y'$, so every neighborhood of x meets Y — in a point other than x, but this fact isn't relevant. Since every neighborhood of x meets Y, an earlier result implies that $x \in \overline{Y}$.

Conversely, suppose that $x \in \overline{Y}$. If $x \in Y$, then $x \in Y \cup Y'$, and I'm done.

Suppose then that $x \notin Y$. Since $x \in \overline{Y}$, every neighborhood of x intersects Y. But the intersection can't consist of x, since $x \notin Y$. Therefore, every neighborhood of x intersects Y in a point other than x. Hence, $x \in Y'$, and certainly $x \in Y \cup Y'$. $\square$

Corollary. Let X be a topological space, and let $Y \subset X$. Then Y is closed if and only if $Y' \subset Y$.

Proof. If Y is closed, then $Y = \overline{Y} = Y \cup Y'$, so $y' \subset Y$.

Conversely, if $Y' \subset Y$, then $Y \cup Y' = Y$, so $\overline{Y} = Y$, and Y is closed. $\square$

The last result is often expressed by saying that *a closed set contains all its limit points*. Thus, you often show a set is closed by taking a limit point and showing that it lies in the set.