7-11-1999

Compact Spaces

In the real line, compactness is equivalent to being closed and bounded. The Heine-Borel Theorem says that a closed and bounded subset B of $\mathbb{R}$ has the property that if $\{U_i\}_{i\in I}$ is a collection of open intervals whose union contains B (i.e., the collection **covers** B), then some finite subcollection of the U's also covers B. This is the appropriate way to generalize "closed and bounded" to arbitary spaces.

Definition.

1. A topological space X is **covered** by a collection $\{U_i\}_{i\in I}$ of subsets if $X = \bigcup_{i\in I} U_i$.
2. If X is a topological space, $Y \subset X$, and $\{U_i\}_{i\in I}$ is a collection of subsets of X, then $\{U_i\}_{i\in I}$ **covers** Y if $Y \subset \bigcup_{i\in I} U_i$.
3. If $\{U_i\}_{i\in I}$ covers X and each U_i is open, then $\{U_i\}_{i\in I}$ is an **open cover** of X.
4. If a subcollection of a covering also covers the space, the subcollection is a **subcovering**.

Example. The collection

$$\{(n, n+1) \mid n \in \mathbb{Z}\} \cup \left\{ \left(n - \frac{1}{2}, n + \frac{1}{2}\right) \;\middle|\; n \in \mathbb{Z} \right\}$$

is an open cover of $\mathbb{R}$.

Every point of $\mathbb{R} - \mathbb{Z}$ is contained in an interval of the form $(n, n+1)$ for some $n \in \mathbb{Z}$. Each integer n is contained in $\left(n - \frac{1}{2}, n + \frac{1}{2}\right)$. Thus, the collection covers $\mathbb{R}$; since each element of the collection is open, it is an open cover.

The collection of open intervals with rational endpoints is also an open cover of $\mathbb{R}$.

More generally, the collection of open balls with rational centers and rational radii forms an open cover of $\mathbb{R}^n$. $\square$

Definition. A space X is **compact** if for every open cover $\{U_i\}_{i\in I}$ of X, some finite subcollection $\{U_1, \ldots, U_n\}$ also covers X.

To say it another way, a space is compact if every covering has a finite subcovering.

It is clear that compactness is preserved by homeomorphisms: If X is compact and X is homeomorphic to Y, then Y is compact.

Example. Any finite set of points (with any topology) is compact, since any open cover can contain at most finitely many (distinct) open sets. $\square$

Example. $\mathbb{R}$ is not compact.

Consider the following open cover of $\mathbb{R}$:

$$\{(n, n+1) \mid n \in \mathbb{Z}\} \cup \left\{ \left(n - \frac{1}{2}, n + \frac{1}{2}\right) \;\middle|\; n \in \mathbb{Z} \right\}.$$

Each integer n is contained in exactly one element of the cover, namely $\left(n - \dfrac{1}{2}, n + \dfrac{1}{2}\right)$. Therefore, any finite subcollection contains at most finitely many integers, so no finite subcollection can cover $\mathbb{R}$.

Since every open interval (a, b) is homeomorphic to $\mathbb{R}$, open intervals aren't compact.

On the other hand, I'll show later that a closed interval $[a, b]$ is compact. $\square$

The following condition is equivalent to compactness; it is extremely important in analysis.

Definition. Let X be a topological space. A collection of subsets $\mathcal{C}$ of X satisfies the **finite intersection condition** if the intersection of any finite subcollection of $\mathcal{C}$ is nonempty.

In other words, if $C_1, \ldots, C_n \in \mathcal{C}$, then $\bigcap_{i=1}^n C_i \neq \emptyset$.

Theorem. Let X be a topological space. X is compact if and only if every collection of closed sets satisfying the finite intersection condition has nonempty intersection.

Proof. Suppose X is compact, and let $\{C_i\}_{i \in I}$ be a collection of closed subsets of X such that every finite subcollection has nonempty intersection. I want to show that $\bigcap_{i \in I} C_i \neq \emptyset$.

Suppose on the contrary that $\bigcap_{i \in I} C_i = \emptyset$. Then

$$X = X - \bigcap_{i \in I} C_i = \bigcup_{i \in I} (X - C_i).$$

Therefore, $\{X - C_i\}_{i \in I}$ is an open cover of X. By compactness, some finite subollection $\{X - C_1, \ldots, X - C_n\}$ covers X. Thus,

$$X = \bigcup_{i=1}^n (X - C_i) = X - \bigcap_{i=1}^n C_i.$$

Hence, $\bigcap_{i=1}^n C_i = \emptyset$, contrary to assumption.

Therefore, $\bigcap_{i \in I} C_i \neq \emptyset$.

Conversely, suppose that every collection of closed subsets of X satisfying the finite intersection condition has nonempty intersection. Let $\{U_i\}_{i \in I}$ be an open cover of X. I want to show that $\{U_i\}_{i \in I}$ has a finite subcover.

Suppose on the contrary that no finite subcollection of $\{U_i\}_{i \in I}$ covers X. Consider the complements $\{X - U_i\}_{i \in I}$. This is a collection of closed sets. If $\{X - U_1, \ldots, X - U_n\}$ is a finite subcollection, then

$$\bigcap_{i=1}^n (X - U_i) = X - \bigcup_{i=1}^n U_i \neq \emptyset.$$

For $X \neq \bigcup_{i=1}^n U_i$, since no finite subcollection of $\{U_i\}_{i \in I}$ covers X.

Thus, every finite subcollection of $\{X - U_i\}_{i \in I}$ has nonempty intersection. By assumption, $\{X - U_i\}_{i \in I}$ has nonempty intersection. Therefore,

$$X - \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X - U_i) \neq \emptyset.$$

This contradicts the fact that $\{U_i\}_{i \in I}$ covers X.

Hence, some finite subcollection of $\{U_i\}_{i \in I}$ covers X, and X is compact. $\square$

Thus, the intersection condition is really just a translation of the definition of compactness from open to closed sets. In any situation, you can choose the version that is easier to apply.

The results that follow describe how compactness behaves in connection with other topological concepts.

Lemma. Let X be a topological space, and let $Y \subset X$. Y is compact if and only if every open cover of Y has a finite subcover.

Reminder: to say that $\{U_i\}_{i \in I}$ is an open cover of Y *as a subset of* X means that each U_i is open in X, and $Y \subset \bigcup_{i \in I} U_i$. That is, the U_i's are subsets of X, and not necessarily subsets of Y.

Proof. Suppose Y is compact. Let $\{U_i\}_{i \in I}$ be an open cover of Y by sets open in X. Then $\{U_i \cap Y\}_{i \in I}$ is an open cover of Y by sets open in Y. By compactness, some finite subcollection $\{U_1 \cap Y, \ldots, U_n \cap Y\}$ covers Y. Then $\{U_1, \ldots, U_n\}$ covers Y.

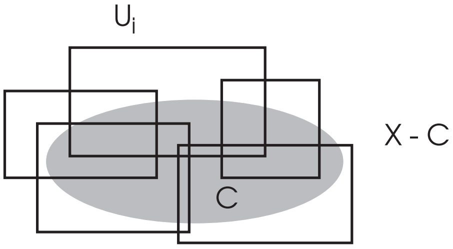
Conversely, suppose that every open cover of Y has a finite subcover. Let $\{U_i\}_{i \in I}$ be an open cover of Y by sets open in Y. For each i, $U_i = Y \cap V_i$, where V_i is open in X.

Now $\{V_i\}_{i \in I}$ is an open cover of Y by sets open in X, so by assumption a finite subcollection $\{V_1, \ldots, V_n\}$ covers Y. Then $\{Y \cap V_1, \ldots, Y \cap V_n\} = \{U_1, \ldots, U_n\}$ covers Y. Therefore, Y is compact. $\square$

One way of putting this is: Compactness is intrinsic to the space. That is, to say Y is *compact* (in the sense of the "open cover" property) has the same meaning whether Y is considered as a subspace of another space or as a space in its own right. In contrast, whether a set is closed or not depends on the topology of the ambient space.

Theorem. A closed subset of a compact space is compact.

Proof. Let X be a compact topological space, and let C be a closed subset of X. Let $\{U_i\}_{i \in I}$ be an open cover of C, and consider the collection $\{U_i\}_{i \in I} \cup \{X - C\}$. This is an open cover of X, so by compactness it has a finite subcover.



If $X - C$ occurs in this subcover, throw it out. The remaining sets $\{U_1, \ldots, U_n\}$ cover C. Thus, I've found a finite subcover of C, so C is compact. $\square$

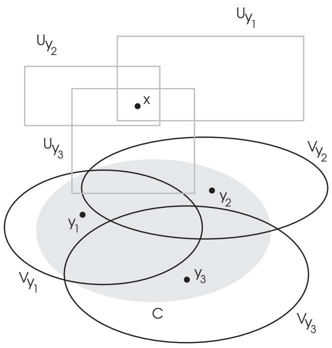
Theorem. A compact subset of a Hausdorff space is closed.

Proof. Let X be a Hausdorff space, and let C be a compact subset of X. I'll show that the complement of C is open.

Let $x \in X - C$. I want to find a neighborhood of x that is contained in $X - C$.

Since X is Hausdorff, for each $y \in C$, there are disjoint neighborhoods U_y of x and V_y of y. Now $\{V_y\}_{y \in Y}$ is an open cover of C, so by compactness there is a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$. Let

$$U = \bigcap_{i=1}^{n} U_{y_i}.$$



If $z \in U$, then $z \in U_{y_i}$ for $i = 1, \ldots, n$. Hence, $z \notin V_{y_i}$ for $i = 1, \ldots, n$. But $\{V_{y_1}, \ldots, V_{y_n}\}$ cover C, so $z \notin C$, i.e. $z \in X - C$. Thus, U is a neighborhood of x contained in $X - C$. Hence, $X - C$ is open, so C is closed. □

Remark. In the course of the proof, I showed that in a Hausdorff space, a compact set (C) and a point in the complement (x) may be separated by disjoint open sets. □

Theorem. The continuous image of a compact space is compact.

Proof. Let X be a compact topological space, let Y be a topological space, and let $f : X \to Y$ be a surjective continuous function. I want to show that Y is compact.

Let $\{U_i\}_{i \in I}$ be an open cover of Y. Then $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of X, so by compactness I may find a finite subcover $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$.

Now if $y \in Y$, then $y = f(x)$ for some $x \in X$. Find $i \in \{1, \ldots, n\}$ such that $x \in f^{-1}(U_i)$. Then $y = f(x) \in U_i$. This shows that $\{U_1, \ldots, U_n\}$ covers Y.

Hence, Y is compact. □

The proof of the following theorem makes nice use of the last three results.

Theorem. Let X and Y be topological spaces, where X is compact and Y is Hausdorff. Let $f : X \to Y$ be a continuous bijection. Then f is a homeomorphism.

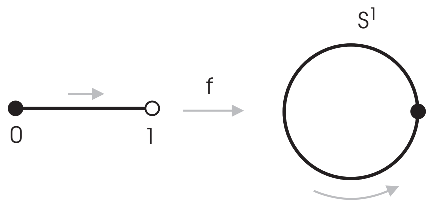
Proof. I need to show that f^{-1} is continuous. I will show that the inverse image of a closed set is closed.

Let C be closed in X; I want to show that $(f^{-1})^{-1}(C) = f(C)$ is closed in Y. Since X is compact and C is closed, C is compact. Since f is continuous and C is compact, $f(C)$ is compact. Since Y is Hausdorff and $f(C)$ is compact, $f(C)$ is closed.

Therefore, f^{-1} is continuous, and f is a homeomorphism. □

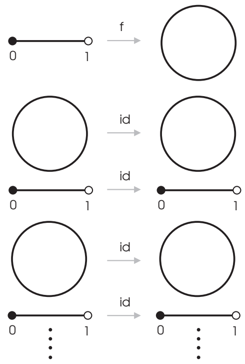
Example. Define $f : [0, 1) \to S^1$ by

$$f(t) = (\cos(2\pi t), \sin(2\pi t)).$$



f is a continuous bijection, and S^1 is surely Hausdorff. However, f is not a homeomorphism. Intuitively, f^{-1} "unwraps" the circle onto the interval; it fails to be continuous at the point $(1, 0)$, where the circle needs to be "cut" in order to do the unwrapping. This is not continuous, because points on either side of the cut point which start off close together wind up far apart, at opposite ends of $[0, 1)$.

Let X be the union of countably many copies of $[0, 1)$ and countably many copies of S^1. Construct a continuous bijection of X to itself by defining the map to be the "wrapping map" f from one of the $[0, 1)$'s to one of the S^1's and the identity maps id : $[0,1) \to [0,1)$ and id : $S^1 \to S^1$ on all the other pieces.

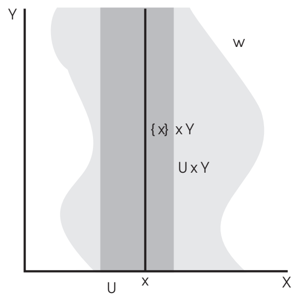


The inverse fails to be continuous for the same reason that f is not continuous.

This gives an example of a continuous bijection from a Hausdorff space to itself which is not a homeomorphism. □

The following result can be used to show that a *finite* product of compact spaces is compact; it's of interest in its own right.

Theorem. (**The Tube Lemma**) Let X and Y be topological spaces, and suppose Y is compact. Let $x \in X$, and let $W \subset X \times Y$ be an open set containing $\{x\} \times Y$. Then there is a neighborhood U of x such that $U \times Y \subset W$.



Proof. For each $y \in Y$, there is a neighborhood $U_y \times V_y$ of (x, y) such that U_y is a neighborhood of x in X, V_y is a neighborhood of y in Y, and $U_y \times V_y \subset W$. It follows that $\{U_y \times V_y\}_{y \in Y}$ is an open cover of $\{x\} \times Y$.

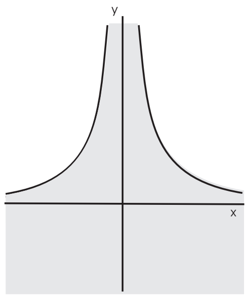
Now $\{x\} \times Y$ is compact, since it's homeomorphic to Y. Therefore, there is a finite subcollection $\{U_{u_1} \times V_{y_1}, \ldots U_{y_n} \times V_{y_n}\}$ which covers $\{x\} \times Y$.

Let $U = \bigcap_{i=1}^{n} U_{y_i}$. U is an open set containing x. If $(z, y) \in U \times Y$, then $y \in V_{y_i}$ for some $i \in \{1, \ldots, n\}$. So $(z, y) \in U \times V_{y_i} \subset W$, and hence $U \times Y \subset W$. □

Example.

$$W = \{(0, y) \mid y \in \mathbb{R}\} \cup \left\{(x, y) \mid y < \left|\frac{1}{x}\right|, \quad x \neq 0\right\}$$

is an open subset of $\mathbb{R}^2$ which contains the y-axis (i.e. $\{0\} \times \mathbb{R}$.



However, there is no open subset U of $\mathbb{R}$ such that

$$\{(0, y) \mid y \in \mathbb{R}\} \subset U \times \mathbb{R} \subset W.$$

The Tube Lemma does not apply because $\mathbb{R}$ is not compact. $\square$

Lemma. Let X be a topological space. Let $\mathcal{D}$ be a collection of subsets of X which is maximal with respect to satisfying the finite intersection condition. Then:

1. $\mathcal{D}$ is closed under finite intersections.

2. If $C \subset X$ and C meets every element of $\mathcal{D}$, then $C \in \mathcal{D}$.

Proof. 1. Suppose $D_1, \ldots, D_n \in \mathcal{D}$. I must show that $D = \bigcap_{i=1}^n D_i \in \mathcal{D}$.

Consider the set $\mathcal{D}' = \mathcal{D} \cup \{D\}$. I claim that $\mathcal{D}'$ satisfies the finite intersection condition.

Take $E_1, \ldots, E_m \in \mathcal{D}'$. If $E_1, \ldots, E_m \in \mathcal{D}$, then $\bigcap_{i=1}^m E_i \neq \emptyset$, since $\mathcal{D}$ satisfies the finite intersection condition.

Otherwise, one of the E's is D; without loss of generality, say $E_1 = D$ and $E_2, \ldots, E_m \in \mathcal{D}$. Then

$$\bigcap_{i=1}^m E_i = D \cap (E_2 \cap \cdots \cap E_m) = \bigcap_{i=1}^n D_i \cap (E_2 \cap \cdots \cap E_m) \neq \emptyset,$$

since this is a finite intersection of elements of $\mathcal{D}$.

Now $\mathcal{D}'$ satisfies the finite intersection condition and it contains $\mathcal{D}$. By maximality of $\mathcal{D}$, $\mathcal{D}' = \mathcal{D}$, which means that $D \in \mathcal{D}$.

2. Suppose $C \subset X$ and C meets every element of $\mathcal{D}$. I want to show that $C \in \mathcal{D}$.

Consider the set $\mathcal{D}' = \mathcal{D} \cup \{C\}$. I claim that $\mathcal{D}'$ satisfies the finite intersection condition.

Take $E_1, \ldots, E_m \in \mathcal{D}'$. If $E_1, \ldots, E_m \in \mathcal{D}$, then $\bigcap_{i=1}^m E_i \neq \emptyset$, since $\mathcal{D}$ satisfies the finite intersection condition.

Otherwise, one of the E's is C; without loss of generality, say $E_1 = C$ and $E_2, \ldots, E_m \in \mathcal{D}$. Now $E_2 \cap \cdots \cap E_m \in \mathcal{D}$ by part 1, so

$$C \cap (E_2 \cap \cdots \cap E_m) \neq \emptyset,$$

since C meets every element of $\mathcal{D}$.

Now $\mathcal{D}'$ satisfies the finite intersection condition and it contains $\mathcal{D}$. By maximality of $\mathcal{D}$, $\mathcal{D}' = \mathcal{D}$, which means that $C \in \mathcal{D}$. $\square$

All of the proofs of Tychonoff's theorem are sophisticated; it's a difficult and important result. Lang [1] credits the following proof to Nicholas Bourbaki, the *nom-de-plume* of a group of mathematicians who wrote an extremely influential series of expository monographs.

Theorem. **(Tychonoff)** Let $\{X_a\}_{a\in A}$ be a family of compact topological spaces. Then $X = \prod_{a\in A} X_a$ is compact.

Proof. Let $\mathcal{C}$ be a collection of closed subsets of X satisfying the finite intersection condition. I need to show that $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$.

Step 1. Consider the set Γ of collections of subsets of X (closed or otherwise) which satisfy the finite intersection condition and contain $\mathcal{C}$. I claim that Γ has maximal elements.

Order Γ by inclusion.
Let Γ' be a chain in Γ. Let

$$\mathcal{F} = \bigcup_{\mathcal{G}\in\Gamma'} \mathcal{G}.$$

Since each $\mathcal{G} \in \Gamma'$ contains $\mathcal{C}$, $\mathcal{F}$ also contains $\mathcal{C}$. I claim that $\mathcal{F}$ satisfies the finite intersection condition.

Let $F_1, \ldots, F_n \in \mathcal{F}$. Suppose $F_1 \in \mathcal{G}_1$, ..., $F_n \in \mathcal{G}_n$, where $\mathcal{G}_1, \ldots, \mathcal{G}_n \in \Gamma'$. Since Γ' is a chain, there is an index $k \in \{1, \ldots, n\}$ such that $F_i \in \mathcal{G}_k$ for $1 \leq i \neq n$. Since $\mathcal{G}_k$ satisfies the finite intersection condition, $F_1 \cap \cdots \cap F_n \neq \emptyset$. This proves that $\mathcal{F}$ satisfies the finite intersection condition.

Since $\mathcal{F}$ satisfies the finite intersection condition and contains $\mathcal{C}$, it's an element of Γ, and it's clearly an upper bound for Γ'. By Zorn's lemma, Γ has maximal elements.

Step 2. Let $\mathcal{D}$ be a maximal element of Γ. I'll construct an element $x \in \bigcap_{D\in\mathcal{D}} D$.

Let $\pi_a : X \to X_a$ be the a^{th} projection map. Fix $a \in A$, and consider the family of sets $\{\pi_a D\}_{D\in\mathcal{D}}$. For any finite subcollection $\{\pi_a(D_1), \ldots, \pi_a(D_n)\}$, the intersection $D_1 \cap \cdots \cap D_n$ is nonempty, because $\mathcal{D}$ satisfies the finite intersection condition. Since

$$\pi_a(D_1 \cap \cdots \cap D_n) \subset \pi_a(D_1) \cap \cdots \cap \pi_a(D_n),$$

it follows that $\pi_a(D_1) \cap \cdots \cap \pi_a(D_n) \neq \emptyset$.

Therefore, $\{\pi_a D\}_{D\in\mathcal{D}}$ satisfies the finite intersection condition.

Hence, $\{\overline{\pi_a D}\}_{D\in\mathcal{D}}$ satisfies the finite intersection condition.

Now $\{\overline{\pi_a D}\}_{D\in\mathcal{D}}$ is a family of closed subsets of the compact space X_a. Therefore, there is an element

$$x_a \in \bigcap_{D\in\mathcal{D}} \overline{\pi_a D}.$$

Define $x = (x_a)_{a\in A}$.

Step 3. Let U_a be a neighborhood of x_a in X_a. I claim that $\pi_a^{-1}(U_a) \in \mathcal{D}$.

Since U_a is a neighborhood of x_a and $x_a \in \overline{\pi_a D}$ for all $D \in \mathcal{D}$, U_a meets $\pi_a(D)$ for all $D \in \mathcal{D}$. But if $U_a \cap \pi_a(D) \neq \emptyset$, then $\pi_a^{-1}(U_a) \cap D \neq \emptyset$. To see this, note that if $z \in U_a \cap \pi_a(D)$, then $z = \pi_a(y)$ for some $y \in D$ and so $\pi_a(y) \in U_a$. Thus, $y \in \pi_a^{-1}(U_a)$, and hence $y \in \pi_a^{-1}(U_a) \cap D$.

Thus, $\pi_a^{-1}(U_a)$ meets every set $D \in \mathcal{D}$. By the second part of the lemma, $\pi_a^{-1}(U_a) \in \mathcal{D}$.

Step 4. Every basic open set containing x is contained in $\mathcal{D}$.

The sets $\pi_a^{-1}(U_a)$ for $a \in A$ and U_a a neighborhood of x_a in X_a form a subbasis for the open sets containing x. A basic open set containing x is a finite intersection of $\pi_a^{-1}(U_a)$-sets; such a finite intersection is in $\mathcal{D}$, by the first part of the lemma.

Step 5. Finally, I'll show that $x \in \bigcap_{C\in\mathcal{C}} C$.

Let $D \in \mathcal{D}$, and let U be a basic open set containing x. Since $\mathcal{D}$ satisfies the finite intersection condition, $U \cap D \neq \emptyset$. Since every basic open set containing x meets D, it follows that $x \in \overline{D}$.

In particular, if D is a *closed* set in $\mathcal{D}$, then $x \in D$. But all the sets in $\mathcal{C}$ are closed, and $\mathcal{C} \subset \mathcal{D}$, so $x \in C$ for all $C \in \mathcal{C}$. Therefore, $x \in \bigcap_{C\in\mathcal{C}} C$.

This completes the proof that X is compact. $\square$

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[1] Serge Lang, *Real analysis* (2^{nd} edition). Reading, Massachusetts: Addison-Wesley Publishing Company, 1983. [ISBN 0-201-14179-5]

