7-11-1999

Compact Sets and the Real Numbers

Theorem. If X is an ordered set satisfying the least upper bound property, then any closed interval $[a, b]$ in X is compact.

Proof. The outline of the proof is as follows. Let $\{U_i\}_{i \in I}$ be an open cover of $[a, b]$. Construct the set C of points $y \in [a, b]$ such that $[a, y]$ can be covered by a finite subcollection of $\{U_i\}_{i \in I}$. Take the least upper bound c of C and show that $c \in C$. Finally, show that $c = b$.

Step 1. Suppose $a \le x < b$. I claim that for some $y > x$, the interval $[x, y]$ can be covered by at most two elements of $\{U_i\}_{i \in I}$.

If x has an immediate successor $x + 1$, then $[x, x + 1]$ has only two elements, so it can be covered by at most two U's.

If x does not have an immediate successor, find U_i containing x. Pick $y' > x$ such that $[x, y') \subset U_i$; this is possible since U_i is open. Since x does not have an immediate successor, there is an element y such that $x < y < y'$. Then $[x, y] \subset U_i$, and $[x, y]$ is covered by a single element of $\{U_i\}_{i \in I}$.

Step 2. Now let

$$C = \{y \in (a, b] \mid [a, y] \text{ can be covered by finitely many } U_i\}.$$

By Step 1, there is an element $y > a$ such that $[a, y]$ can be covered by at most two elements of $\{U_i\}_{i \in I}$. Therefore, C is nonempty. Let c be the least upper bound of C in $[a, b]$.

Step 3. I claim that $c \in C$.

Find U_i containing c. U_i is open and $c > a$, so I may find an interval $(d, c] \subset U_i$. Since d can't be an upper bound for C, there is an element of C larger than d. Let $c' \in C$, where $d < c' < c$. Then $[a, c']$ can be covered by finitely many U's, and $[c', c] \subset U_i$. Therefore, $[a, c] = [a, c'] \cup [c', c]$ can be covered by finitely many U's. Hence, $c \in C$.

Step 4. I claim that $c = b$.

Suppose that $c < b$. By Step 1, there is a $y > c$ such that $[c, y]$ can be covered by at most two elements of $\{U_i\}_{i \in I}$. Since $c \in C$, $[a, c]$ can be covered by finitely many elements of $\{U_i\}_{i \in I}$. So $[a, y] = [a, c] \cup [c, y]$ can be covered by finitely many elements of $\{U_i\}_{i \in I}$, and therefore $y \in C$. This contradicts the fact that c was the least upper bound of C. Hence, $c = b$.

Since $b \in C$, $[a, b]$ can be covered by finitely many elements of $\{U_i\}_{i \in I}$. Therefore, $[a, b]$ is compact. $\square$

Corollary. A closed interval $[a, b]$ in $\mathbb{R}$ is compact. $\square$

Theorem. (**Heine-Borel**) A subset $C \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded relative to the standard metric or the square metric.

Remark. Recall that the standard metric d and the square metric ρ are related by

$$\rho(x, y) \le d(x, y) \le \sqrt{n} \cdot \rho(x, y).$$

Hence, a set bounded relative to one metric is bounded relative to the other.

Proof. Suppose that C is compact. $\mathbb{R}^n$ is Hausdorff, so C is closed. The collection of balls $\{B(0; n) \mid n \in \mathbb{Z}^+\}$ is an open cover of $\mathbb{R}^n$, so it is an open cover of C. By compactness, there is a finite subcover; the element of the subcover with the largest radius contains C, so C is bounded.

Conversely, suppose C is a closed and bounded subset of $\mathbb{R}^n$. Suppose that $\rho(x, y) \le s$ for all $x, y \in C$. Fix $x \in C$ and let $t = \rho(x, 0)$. Then for all $y \in C$,

$$d(y, 0) \le d(y, x) + d(x, 0) = s + t.$$

It follows that $C \subset [-(s+t), s+t]^n$. However, $[-(s+t), s+t]^n$ is a product of compact spaces, so it's compact. Thus, C is a closed subset of the compact set $[-(s+t), s+t]^n$, so it is compact. $\square$

Theorem. Let X be a compact topological space, let Y be an ordered set with the order topology, and let $f : X \to Y$ be continuous. There are points $a, b \in X$ such that

$$f(a) \leq f(x) \leq f(b) \quad \text{for all} \quad x \in X.$$

Proof. $f(X)$ is a compact subset of Y. I'll show that $f(X)$ has a largest element and a smallest element.

Suppose that $f(X)$ does not have a largest element. Then the collection of open rays

$$\{(-\infty, f(x)) \mid x \in X\}$$

cover $f(X)$. By compactness, I may find $x_1, \ldots, x_n \in X$ such that

$$\{(\infty, f(x_1)), \ldots, (-\infty, f(x_n))\}$$

cover $f(X)$.

Find $m \in \{1, \ldots, n\}$ such that

$$f(x_m) = \max\{f(x_i) \mid 1 \leq i \leq n\}.$$

Then $f(X) \subset (-\infty, f(x_m))$, but $f(x_m) \notin (-\infty, f(x_m))$. This contradiction shows that $f(X)$ has a largest element; a similar argument shows that $f(X)$ has a smallest element. $\square$

In the case where $X = [a, b]$ is a closed interval in $\mathbb{R}$ and $Y = \mathbb{R}$, this is the familiar result from calculus which says that a continuous function on a closed interval has a max and a min on the interval.

The next result and its corollary are more amusing than important, since you've probably seen the Cantor diagonalization proof of the uncountability of $[0, l]$. But this topological proof does not make any reference to *representations* of real numbers as infinite decimal.

Theorem. Let X be a (nonempty) compact Hausdorff space. Suppose every point of X is a limit point of X. Then X is uncountable.

Proof.

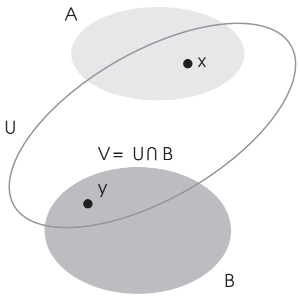
Step 1. Let U be a nonempty open subset of X, and let $x \in X$. I'll construct an open set $V \subset U$ such that $x \notin \overline{V}$.

If $U = \{x\}$, then U is a neighborhood of x which contains only x. This contradicts the assumption that every point of X is a limit point of X. Now x may not be in U in the first place, but this shows that in any event, U can't consist of x alone.

Thus, I may choose $y \in U$ such that $y \neq x$. Since X is Hausdorff, there are disjoint neighborhoods A of x and B of y. Let $V = B \cap U$. Now

$$V = B \cap U \subset B \subset X - A,$$

and $X - A$ is closed. Therefore, $\overline{V} \subset X - A$. In particular, since $x \in A$, $\overline{V}$ does not contain x.



Step 2. Next, I'll show that there is no surjection $f : \mathbb{Z}^+ \to X$.

Suppose that $f : \mathbb{Z}^+ \to X$. For the purposes of numbering, take $V_0 = X$. By Step 1, I may find a nonempty open set V_1 such that $f(1) \notin \overline{V_1}$.

Suppose inductively that $n > 1$, and I have found a nonempty open set $V_{n-1} \subset V_{n-2}$ such that $f(n-1) \notin \overline{V_{n-1}}$. Use Step 1 to find a nonempty open set $V_n \subset V_{n-1}$ such that $f(n) \notin \overline{V_n}$.

Consider the collection $\{\overline{V_n}\}_{n \in \mathbb{Z}^+}$. The collection has the finite intersection condition, since

$$\overline{V_1} \supset \overline{V_2} \supset \overline{V_3} \supset \ldots.$$

By compactness, there is a point $x \in \bigcap_{n \in \mathbb{Z}^+} \overline{V_n}$. Then $x \neq f(n)$ for any $n \in \mathbb{Z}^+$, so f is not surjective.

Since there is no surjective function from $\mathbb{Z}^+$ to X, X is uncountable. $\square$

Corollary. The interval $[0, 1]$ in $\mathbb{R}$ is uncountable. $\square$

Obviously, it follows that $\mathbb{R}$ and open intervals in $\mathbb{R}$ are uncountable.