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Connected Spaces

Definition. A space is **connected** if it cannot be written as the disjoint union of two nonempty open sets.

A **separation** of a topological space X consists of two disjoint nonempty open sets U and V such that $X = U \cup V$.

Thus, a connected space is one that cannot be separated.

Example. If X is connected and Y is homeomorphic to X, then Y is connected. $\Box$

Example. A point is connected (considered as a subspace of an arbitrary topological space). $\Box$

Example. $\mathbb{R}$ is connected in the standard topology.

On the other hand, $\mathbb{Z}$ is not connected when considered as a subspace of $\mathbb{R}$. (The subspace topology is the discrete topology in this case.)

More generally, a set with more than one point is not connected in the discrete topology. $\Box$

Example. If $X = U \cup V$ is a separation of X, then U and V are both open and closed.

Conversely, if A is a subset of a topological space X, $A \neq \emptyset, X$, and A is both open and closed, then $X = A \cup (X - A)$ is a separation of X.

To put it another way, a space is connected if and only if there is no proper nontrivial subset which is both open and closed.

For example, any (nontrivial) partition topology is not connected. $\Box$

Proposition. Let $X = U \cup V$ be a separation of X. If $Y \subset X$ is connected, then $Y \subset U$ or $Y \subset V$.

Proof. Suppose on the contrary that $Y \cap U \neq \emptyset$ and $Y \cap V \neq \emptyset$. Then $Y \cap U$ and $Y \cap V$ are nonempty open subsets of V whose union is V, contradicting the fact that Y is connected. $\Box$

If a space X is not connected, what are the biggest connected subspaces of X?

Lemma. Let $\{U_a\}_{a\in A}$ be a collection of connected subsets of a topological space X. Suppose that $\bigcap_{a\in A} U_a \neq \emptyset$. Then $U = \bigcup_{a\in A} U_a$ is connected.

Proof. Suppose on the contrary that $U = V \cup W$ is a separation of U. Let $x \in \bigcap_{a\in A} U_a$. Then $x \in V$ or $x \in W$; without loss of generality, suppose $x \in V$.

For each a, the connected set U_a is contained in either V or W, by the preceding result. But $x \in U_a$ for all a, so all the U_a's meet V. Therefore, $U_a \subset V$ for all a, and hence $U \subset V$.

This means that $W = \emptyset$, which contradicts the fact that V and W separated U.

Therefore, U is connected. $\Box$

Theorem. Let X be a topological space.

1. Every point $x \in X$ is contained in a unique maximal connected subset C_x of X.

2. $C_x = C_y$ or $C_x \cap C_y = \emptyset$.

To say that C_x is maximal means that C_x is not properly contained in any other connected subset of X.

Proof. 1. Let $x \in X$. Consider the collection $\mathcal{C}$ of connected subsets of X which contain x. $\mathcal{C}$ is nonempty, since $\{x\} \in C$. Order C by inclusion. I'll use Zorn's Lemma to show that $\mathcal{C}$ has maximal elements.

Let $\mathcal{D}$ be a chain in $\mathcal{C}$. Then every $U \in \mathcal{D}$ contains x, so $V = \bigcup_{U \in \mathcal{D}} U$ is a union of connected sets with a point in common. By the preceding result, V is connected; since it contains x and contains every $U \in \mathcal{D}$, it is an upper bound for $\mathcal{D}$ in $\mathcal{C}$.

By Zorn's Lemma, $\mathcal{C}$ has maximal elements.

Next, I'll show that $\mathcal{C}$ has a *unique* maximal element.

If U and V are maximal elements of C, then $U \cup V$ is a connected set, since U and V are connected and have x in common. Thus, $U \cup V$ is a connected set containing x, so it's an element of $\mathcal{C}$. Since U and V are maximal and are contained in $U \cup V$, this is only possible if $U = V$.

Note that if C_x is the unique maximal element in $\mathcal{C}$, any connected set containing C_x would be a connected set containing x — hence, an element of $\mathcal{C}$ containing the maximal element C_x. Therefore, there is no connected set which *properly* contains C_x.

2. Suppose $C_x \cap C_y \neq \emptyset$. Let $z \in C_x \cap C_y$. Then $C_x \cup C_y$ is connected, since it's the union of connected sets having a point in common. Moreover, $C_x \cup C_y$ contains the maximal sets C_x and C_y. By part 1, this is impossible unless $C_x = C_y$. $\square$

The maximal connected sets described by the theorem form a partition of X. They are called the **connected components** of X. Here is another sense in which they are maximal.

Lemma. Let X be a topological space. Every connected subset of X is contained in a component of X.

Proof. Let U be a connected subset of X. U must intersect some component of X; suppose $U \cap C \neq \emptyset$, where U is a component. Then $U \cup C$ is a union of connected sets with nonempty intersection, so it's connected. Since $C \subset U \cup C$, and since C is maximal, I must have $C = U \cup C$. Therefore, $U \subset C$. $\square$

Theorem. The continuous image of a connected set is connected.

Proof. Let $f : X \to Y$ be continuous and surjective, and suppose X is connected. I want to show that Y is connected.

Suppose on the contrary that $Y = U \coprod V$ is a separation of Y. Since U and V are open, $f^{-1}(U)$ and $f^{-1}(V)$ are open; since U and V are disjoint, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint. Therefore, $X = f^{-1}(U) \coprod f^{-1}(V)$ is a separation, contradicting the fact that X is connected. $\square$

The theorem will be useful once I have some connected spaces to play with.

I'm working toward showing that the product of connected spaces is connected; the preliminaries are interesting in their own right.

Lemma. Let X be a topological space, let $Y \subset X$, and let $Y = A \coprod B$ be a partition of Y. A and B separate Y if and only if neither A nor B contains an X-limit point of the other.

Proof. Suppose A and B separate Y. Then A and B are closed in Y, so

$$\overline{A} \cap Y = \operatorname{cl}_Y A = A.$$

Suppose x is an X-limit point of A, so $x \in \overline{A}$. If $x \in B$, then $x \in Y$, so $x \in \overline{A} \cap Y = A$. But then $x \in A \cap B$, contradicting the fact that A and B are disjoint.

Therefore, B contains no X-limit points of A.

A similar argument shows that A contains to X-limit points of B.

Conversely, suppose neither A nor B contains an X-limit point of the other. Let A' denote the X-limit points of A. Then $A' \cap Y$ is the set of X-limit points of A contained in Y; by assumption, none of these are in B, so any such points must be in A. That is, $A' \cap Y \subset A$. Then

$$\overline{A} \cap Y = (A \cup A') \cap Y = (A \cap Y) \cup (A' \cap Y) \subset A \cup A = A \subset \overline{A} \cap Y.$$

Therefore, $A = \overline{A} \cap Y$, so A is closed in Y.
A similar argument shows that B is closed in Y.
Since A and B partition Y, they form a separation of Y. $\square$

Lemma. If A is connected and

$$A \subset B \subset \overline{A},$$

then B is connected.

Heuristically, if you add some limit points of a connected set to the set, you get a connected set.

Proof. Suppose $B = U \coprod V$ is a separation of B. Thus, U and V are disjoint nonempty open subsets of B. Since A is connected, A lies entirely in U or V. Without loss of generality, say $A \subset U$. Taking closures in X yields $\overline{A} \subset \overline{U}$.

U and V are disjoint, and by the preceding lemma V contains no limit points of U. Thus, $\overline{A} \cap V = \emptyset$. But $B \subset \overline{A}$, so B doesn't intersect V, either. This means that V is empty, which is a contradiction.

Therefore, B is connected. $\square$

Corollary. The closure of a connected set is connected. $\square$

Corollary. Connected components of a space are closed.

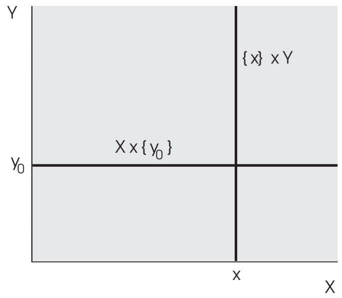
Proof. If C is a connected component of X, then $\overline{C}$ is a connected set containing C, by the previous corollary. By maximality, $C = \overline{C}$, so C is closed. $\square$

Lemma. The product of finitely many connected spaces is connected.

Proof. It suffices to prove this for two spaces, since the general result will follow by induction.

Thus, suppose X and Y are connected. I want to show that $X \times Y$ is connected. The idea is to use the fact that a union of connected sets with a point in common is connected.

To do this, picture $X \times Y$ as a plane. I'll decompose it as the union of "crosses": each "cross" consists of the union of a fixed "horizontal line" with an arbitrary "vertical line".



Fix $y_0 \in Y$. For each $x \in X$, consider the set

$$C_x = (X \times \{y_0\}) \cup (\{x\} \times Y).$$

$X \times \{y_0\}$ is connected, since it's a homeomorphic copy of X; $\{x\} \times Y$ is connected, since it's a homeomorphic copy of Y. The sets have the point (x, y_0) in common. Therefore, C_x is connected.

Next, the union of the C_x's is connected, because it's a union of connected sets having $X \times \{y_0\}$ in common. But the union of the C_x's is $X \times Y$, so $X \times Y$ is connected. $\square$

Theorem. An arbitrary product of connected spaces is connected.

Proof. First, I'll choose a point in the product. I'll construct a union of connected sets having this point in common; the earlier result on unions shows that the union is connected. Finally, I'll show that the closure of the union is the whole product; this will prove that the product is connected, by the earlier result on closures.

Let $X = \prod_{a \in A} X_a$ be a product of connected spaces. I want to show that X is connected. Let $y = (y_a)_{a \in A} \in X$.

For each finite collection $\{a_1, \ldots, a_n\}$ of indices from A, define

$$X(a_1, \ldots, a_n) = (x_a)_{a \in A} \mid x_a = y_a \quad \text{if} \quad a \neq a_1, \ldots, a_n\}.$$

Thus, a point of $X(a_1, \ldots, a_n)$ agrees with y except possibly in the $a_1, \ldots, a_n$ positions.

(Note that this doesn't mean that such an x *doesn't agree* with y in the $a_1, \ldots, a_n$ positions; it means only that such an x *does agree* with y in the other positions.)

I claim that $X(a_1, \ldots, a_n)$ is connected. I'll do this by showing that it's homeomorphic to a finite product of X_a's, which is connected by the preceding lemma.

Define $\phi : X_{a_1} \times \cdots \times X_{a_n} \to X(a_1, \ldots, a_n)$ by

$$[\phi(x_{a_1}, \ldots, x_{a_n})]_a = \begin{cases} x_{a_i} & \text{if } a = a_i \text{ for some } i \\ y_a & \text{if } a \neq a_i \text{ for any } i \end{cases}.$$

Thus, ϕ plugs $x_{a_1}, \ldots, x_{a_n}$ into the $a_1, \ldots, a_n$ components and uses y for the other components.

Define $\psi : X(a_1, \ldots, a_n) \to X_{a_1} \times \cdots \times X_{a_n}$ by

$$\psi[(x_a)] = (x_{a_1}, \ldots, x_{a_n}).$$

ϕ and ψ are easily seen to be inverses, so ϕ is bijective.

A basis element for the product topology on $X_{a_1} \times \cdots \times X_{a_n}$ is a product of open sets from the factors. A basis element for the subspace topology on $X(a_1, \ldots, a_n)$ consists of the intersection of $X(a_1, \ldots, a_n)$ with a basis element for the product $X = \prod_{a \in A} X_a$. A basis element for $X = \prod_{a \in A} X_a$ consists of a product of open sets from the factors, where at most finitely many are not the whole space.

With this description in mind, it is clear that ϕ carries a basis element for $X_{a_1} \times \cdots \times X_{a_n}$ onto a basis element for $X(a_1, \ldots, a_n)$, and ψ carries a basis element for $X(a_1, \ldots, a_n)$ onto a basis element for $X_{a_1} \times \cdots \times X_{a_n}$. Therefore, ϕ is a homeomorphism.

Since $X_{a_1} \times \cdots \times X_{a_n}$ is a finite product of connected spaces, it's connected. Therefore, $X(a_1, \ldots, a_n)$ is connected.

Since $y \in X(a_1, \ldots, a_n)$ for any finite subset $\{a_1, \ldots, a_n\} \subset A$, the union

$$X' = \bigcup \left\{ X(a_1, \ldots, a_n) \,\middle|\, \{a_1, \ldots, a_n\} \subset A \right\}$$

is a union of connected sets with a point in common, hence connected.

Finally, I claim that $\overline{X'} = X$. Let $x \in X$, and let $\prod_{a \in A} U_a$ be a neighborhood of x, where U_a is open in X_a and $U_a = X_a$ for all but finitely many a's. Let $a_1, \ldots, a_n$ be the indices for which $U_a \neq X_a$. Define $z = (z_a)_{a \in A}$ by

$$z_a = \begin{cases} x_{a_i} & \text{if } a = a_i \text{ for some } i \\ y_a & \text{if } a \neq a_i \text{ for any } i \end{cases}.$$

Then $z \in \prod_{a \in A} U_a$ and $z \in X(a_1, \ldots, a_n) \subset X'$.

Thus, every (basic) neighborhood of x intersects X'. Since $x \in X$ was arbitrary, it follows that $\overline{X'} = X$. Since X is the closure of a connected set, it's connected. □