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Continuous Functions

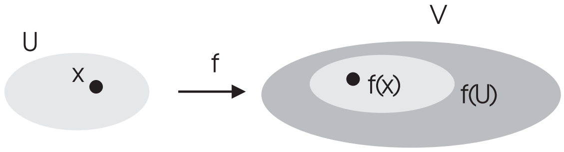
For a function $f : \mathbb{R} \to \mathbb{R}$, f is **continuous** at $x = a$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that if

$$\delta > |x - a| > 0, \quad \text{then} \quad \epsilon > |f(x) - f(a)|.$$

Interpreted in terms of intervals — i.e. basic open sets in $\mathbb{R}$ — this says that for any interval I containing $f(a)$, there is an interval J containing a, such that f maps J into I.

This motivates the following definition.

Definition. Let $f : X \to Y$ be a function, where X and Y are topological spaces. f is **continuous** at $x \in X$ if for every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.



f is **continuous** (on X) if it is continuous at every point of X.

The definition I've given is often called the *pointwise definition* of continuity. I'll give the equivalent *global form* shortly.

Example. Any function $f : \mathbb{R}^n \to \mathbb{R}^m$ that is continuous in the ϵ-δ sense is continuous according to the definition above. $\square$

Example. Let X be a topological space. The identity function $\text{id} : X \to X$ is continuous.

To prove this, let $x \in X$ and let V be a neighborhood of $\text{id}(x) = x$. Then V is a neighborhood of x which satisfies $\text{id}(V) \subset V$. $\square$

Example. Let X and Y be topological spaces, and let $y \in Y$. The constant function $c_y : X \to Y$ defined by $c_y(x) = y$ for all $x \in X$ is continuous.

To see this, suppose $x \in X$ and V is a neighborhood of $c_y(x) = y$. X is a neighborhood of x, and $c_y(X) = \{y\} \subset V$. Therefore, c_y is continuous at x. $\square$

Example. Let Y be a subspace of the topological space X. The inclusion map $i : Y \hookrightarrow X$ given by $i(y) = y$ for all $y \in Y$ is continuous.

To prove this, let $y \in Y$ and let V be a neighborhood of $i(y) = y$ in X. Since V is open in X, $V \cap Y$ is open in Y, and $y \in V \cap Y$. Clearly, $i(V \cap Y) \subset V$, so $V \cap Y$ is a neighborhood of y in Y which i maps into V. Therefore, $\subset$ is continuous at y. $\square$

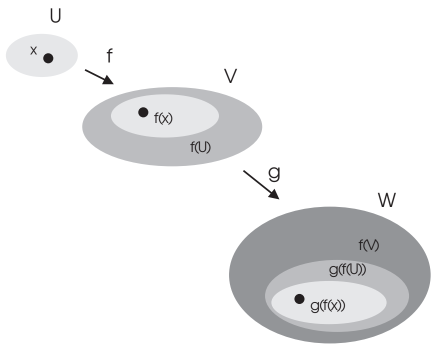
Example. The composite of continuous functions is continuous.

To be specific, suppose that X, Y, and Z are topological spaces, $f : X \to Y$ is continuous at x, and $g : Y \to Z$ is continuous at $f(x)$. Then $g \cdot f : X \to Z$ is continuous at x.

Thus, if f is continuous on X and g is continuous on Y, then $g \cdot f$ is continuous on X.

To prove the result at a point, assume that $f : X \to Y$ is continuous at x and $g : Y \to Z$ is continuous at $f(x)$. Let W be a neighborhood of $g(f(x))$ in Z. By continuity of g at $f(x)$, there is a neighborhood V of $f(x)$ such that $g(V) \subset W$.

By continuity of f at x, there is a neighborhood U of x such that $f(U) \subset V$.



Then $g(f(U)) \subset g(V) \subset W$. so $g \cdot f$ is continuous at x. $\square$

Example. The restriction of a continuous function is continuous.

Specifically, let $f : X \to Y$ be a continuous function between topological spaces. Let U be a subspace of X. Then $f|U : U \to Y$ defined by $f|U(x) = f(x)$ for all $x \in U$ is continuous.

The result is immediate from the last two examples, since $f|U = f \cdot i$, where $i : U \hookrightarrow X$ is the inclusion map, and the composite of continuous functions is continuous. $\square$

Example. If $f : X \to Y$ is a continuous function between topological spaces and Y is a subspace of Z, then the function $\tilde{f} : X \to Z$ defined by $\tilde{f}(x) = f(x)$ for all $x \in X$ is continuous. ($\tilde{f}$ is said to be obtained by **expanding the range**.)

This result is also easy, since $\tilde{f} = i \cdot f$, where $i : Y \hookrightarrow Z$ is the inclusion map and the composite of continuous functions is continuous. $\square$

Next, I'll show that the pointwise definition of continuity is equivalent to the following global definition.

Proposition. Let $f : X \to Y$ be a function between topological spaces. f is continuous if and only if for every open set V in Y, $f^{-1}(V)$ is open in X.

This is expressed more concisely by saying that *the inverse image of an open set is open.*

Proof. Suppose f is continuous, and let V be open in Y. I want to show that $f^{-1}(V)$ is open in X.

Let $x \in f^{-1}(V)$. I want to find a neighborhood of x contained in $f^{-1}(V)$. Now $f(x) \in V$, so by continuity, I can find a neighborhood U of x such that $f(U) \subset V$. Then $U \subset f^{-1}(V)$, and U is the desired neighborhood of x. This proves that $f^{-1}(V)$ is open.

Conversely, suppose that the inverse image of an open set is open. I want to show that f is continuous. Let $x \in X$, and let V be a neighborhood of $f(x)$. I want to find a neighborhood U of x such that $f(U) \subset V$.

By assumption, $f^{-1}(V)$ is open. Since $f(x) \in V$, it follows that $x \in f^{-1}(V)$. Moreover, $f(f^{-1}(V)) \subset V$. Thus, $f^{-1}(V)$ is a neighborhood of x that f maps into V. Hence, f is continuous. $\square$

I believe that the name of the next result is due to James R. Munkres; it's an enormously useful tool for constructing continuous functions.

Lemma. (**The Pasting Lemma**) Let X and Y be topological spaces, and let U and V be open sets in X

such that $X = U \cup V$. Suppose that $f : U \to Y$ and $g : V \to Y$ are continuous, and

$$f(x) = g(x) \quad \text{for all} \quad x \in U \cap V.$$

Then the function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in U \\ g(x) & \text{if } x \in V \end{cases}$$

is continuous.

Proof. Let W be open in Y. Then

$$h^{-1}(W) = f^{-1}(W) \cup g^{-1}(W).$$

By continuity, $f^{-1}(W)$ is open in U; since U is open in X, $f^{-1}(W)$ is open in X.
By continuity, $g^{-1}(W)$ is open in V; since V is open in X, $g^{-1}(W)$ is open in X.
Therefore, $f^{-1}(W) \cup g^{-1}(W)$ is open in X, so $h^{-1}(W)$ is open in X. By the preceding result, h is continuous.

An analogous result holds if U and V are closed; the proof follows from the next result, which says that you can also express continuity in terms of closed sets.

Proposition. Let $f : X \to Y$ be a function between topological spaces. The following statements are equivalent:

1. f is continuous.
2. For all $U \subset X$, $f(\overline{U}) \subset \overline{f(U)}$.
3. The inverse image of a closed set is closed.

Proof. $(1 \Rightarrow 2)$ Suppose f is continuous, and let $U \subset X$. Since $\overline{U} = U \cup U'$, I have

$$f(\overline{U}) = f(U \cup U') = f(U) \cup f(U').$$

Clearly, $f(U) \subset \overline{f(U)}$; I need to show that $f(U') \subset \overline{f(U)}$.
Let x be a limit point of U, and let V be a neighborhood of $f(x)$. I must show that V intersects $f(U)$. Since $f^{-1}(V)$ is a neighborhood of x and x is a limit point of U, there is a $y \in f^{-1}(V) \cap U$ such that $y \neq x$. Then

$$f(y) \in f\left(f^{-1}(V) \cap U\right) \subset f(f^{-1}(V)) \cap f(U) \subset V \cap f(U).$$

This proves that V intersects $f(U)$, so $f(x) \in \overline{f(U)}$. Thus, $f(U') \subset \overline{f(U)}$, and so $f(U) \cup f(U') \subset \overline{f(U)}$.
Therefore, $f(\overline{U}) \subset \overline{f(U)}$.

$(2 \Rightarrow 3)$ Suppose that for all $U \subset X$, $f(\overline{U}) \subset \overline{f(U)}$. I want to show that the inverse image of a closed set is closed.
Let C be closed in Y. I want to show that $f^{-1}(C)$ is closed in X. Now

$$f\left(\overline{f^{-1}(C)}\right) \subset \overline{f\left(f^{-1}(C)\right)} \subset \overline{C} = C,$$

where the first inclusion follows from statement 2, the second inclusion follows from $f\left(f^{-1}(C)\right) \subset C$, and the last equality follows from the fact that C is closed.
Thus, $\overline{f^{-1}(C)} \subset f^{-1}(C)$. But clearly, $f^{-1}(C) \subset \overline{f^{-1}(C)}$, so $\overline{f^{-1}(C)} = f^{-1}(C)$, and $f^{-1}(C)$ is closed.

$(3 \Rightarrow 1)$ Suppose the inverse image of a closed set is closed. I want to show that f is continuous.
Let V be an open subset of Y. Then $Y - V$ is closed, so by assumption $f^{-1}(Y - V)$ is closed. But

$$f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V).$$

Therefore, $X - f^{-1}(V)$ is closed, so $f^{-1}(V)$ is open. Therefore, f is continuous. □