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The Countability Axioms

Definition. Let X be a topological space, and let $x \in X$.

1. X has a **countable basis at** x if there is a countable collection of neighborhoods $\{B_i\}_{i \in I}$ of x such that if U is a neighborhood of x, then $B_i \subset U$ for some i.
2. X is **first countable** if X has a countable basis at each point.
3. X is **second countable** if X has a countable basis for its topology.

   Obviously, a second countable space is first countable.

Example. Every metric space is first countable. If X is a metric space and $x \in X$, then $\left\{ B\left(x; \dfrac{1}{n}\right) \;\middle|\; n \in \mathbb{Z}^+ \right\}$ is a countable basis at x. $\square$

Example. $\mathbb{R}^n$ is second countable, since the balls with rational centers and radii form a countable basis for the usual topology.

On the other hand, $\mathbb{R}$ with the discrete topology is not second countable. $\square$

Example. Consider $\mathbb{R}$ with the finite complement topology. Thus, the open sets are $\emptyset$ and any set whose complement is finite.

I claim that $\mathbb{R}$ does not have a countable basis at any point. Suppose, for example, that there is a countable basis $\{B_1, B_2, \ldots\}$ at 0. I claim that $\bigcap_{i=1}^{\infty} B_i = \{0\}$. Clearly, 0 is in the intersection; I must show that no nonzero point is in the intersection.

Suppose that $x \neq 0$. The complement of $\mathbb{R} - \{x\}$ is finite, so it's an open set containing 0. Hence, $B_i \subset \mathbb{R} - \{x\}$ for some i. Then $x \notin B_i$, so $x \notin \bigcap_{i=1}^{\infty} B_i$.

Thus,

$$\mathbb{R} - \{0\} = \mathbb{R} - \bigcap_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} (\mathbb{R} - B_i).$$

Each set B_i is finite, so $\bigcup_{i=1}^{\infty} (\mathbb{R} - B_i)$ is countable. But $\mathbb{R}$ is uncountable, so $\mathbb{R} - \{0\}$ is uncountable. This contradiction shows that there is no countable basis at 0; obviously, the argument works for any $x \in \mathbb{R}$.

Thus, $\mathbb{R}$ is not first countable (or second countable) in the finite complement topology. $\square$

The following results were proved for metric spaces. An examination of the proofs shows that they only depended on the fact that metric spaces are first countable.

Theorem. Let X and Y be topological spaces, and suppose X is first countable.

1. Let $A \subset X$. $x \in \overline{A}$ if and only if there is a sequence of points of A converging to x.
2. Let $f : X \to Y$. f is continuous if and only if whenever $\{x_n\}$ is a convergent sequence in X, $\{f(x_n)\}$ is a convergent sequence in Y. $\square$

Proposition.

1. A subspace of a first countable space is first countable.

2. A countable product of first countable spaces is first countable.

3. A subspace of a second countable space is second countable.

4. A countable product of second countable spaces is second countable.

Proof. 1. Suppose X is first countable, and suppose Y is a subspace of X. Let $y \in Y$. Let $\{B_1, B_2, \ldots\}$ be a countable basis at y in X. Then $\{B_1 \cap Y, B_2 \cap Y, \ldots\}$ is a countable basis at y in Y.

2. Suppose $X_1, X_2, \ldots$ are first countable spaces. Let $(x_n) \in \prod_{n=1}^{\infty} X_n$. Let $\mathcal{B}_n$ be a countable basis for x_n in X_n. Consider the collection $\mathcal{U}$ of product neighborhoods $\prod_{n=1}^{\infty} U_n$, where $U_n = X_n$ for all but finitely many n, and if $U_n \neq X_n$, then $U_n \in \mathcal{B}_n$. Then $\mathcal{U}$ is a countable basis at (x_n) in $\prod_{n=1}^{\infty} X_n$, so $\prod_{n=1}^{\infty} X_n$ is first countable.

3. Suppose X is second countable, and suppose Y is a subspace of X. Let $\{B_1, B_2, \ldots\}$ be a countable basis for the topology on X. Then $\{B_1 \cap Y, B_2 \cap Y, \ldots\}$ is a countable basis for the subspace topology on Y.

4. Suppose $X_1, X_2, \ldots$ are second countable spaces. Let $\mathcal{B}_n$ be a countable basis for the topology on X_n.Consider the collection $\mathcal{U}$ of product neighborhoods $\prod_{n=1}^{\infty} U_n$, where $U_n = X_n$ for all but finitely many n, and if $U_n \neq X_n$, then $U_n \in \mathcal{B}_n$. Then $\mathcal{U}$ is a countable basis for the product topology on $\prod_{n=1}^{\infty} X_n$, so $\prod_{n=1}^{\infty} X_n$ is second countable. $\square$

Definition. Let X be a topological space.

1. X is **Lindelöf** if every open cover of X has a countable subcover.

2. X is **separable** if X has a countable dense subset.

Example. Any compact space is Lindelöf, since every open cover has a *finite* subcover. $\square$

Example. $\mathbb{R}$ is separable, since $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. $\square$

Proposition. Let X be a second countable topological space.

1. X is Lindelöf.

2. X is separable.

Proof. 1. Let $\mathcal{U}$ be an open cover of X. Let $\{B_1, B_2, \ldots\}$ be a countable basis for X. For each $n \in \mathbb{Z}^+$, let

$$\mathcal{U}_n = \{U \in \mathcal{U} \mid B_n \subset U\}.$$

Next, for each nonempty $\mathcal{U}_n$, choose $V_n \in \mathcal{U}_n$.

For each $x \in X$, there is a $U \in \mathcal{U}$ such that $x \in U$. Moreover, there is a basis element B_n such that $x \in B_n \subset U$. Then $\mathcal{U}_n$ is nonempty, so V_n is defined, and $x \in B_n \subset V_n$.

Now x was an arbitrary point of X, so the collection $\{V_n\}$ covers X. $\{V_n\}$ is a countable subcover of $\mathcal{U}$.

2. Let $\{B_1, B_2, \ldots\}$ be a countable basis for the topology of X. Choose $x_n \in B_n$ for each $n \in \mathbb{Z}^+$. Then $\{x_n\}$ is a countable dense subset of X. $\square$

