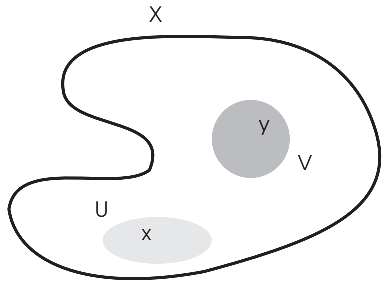
6-17-1999

Hausdorff Spaces

If you try to study topological spaces without any conditions, the variety becomes overwhelming — there is little you can say about topological spaces *in general*. In order to say more, you have to assume more. The **Hausdorff** condition is a common assumption mathematicians make about topological spaces.

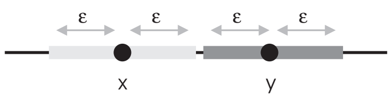
Definition. A topological space X is **Hausdorff** (or T_2)if for all $x, y \in X$, $x \neq y$, there are neighborhoods U of x and V of y such that $U \cap V = \emptyset$.



Remark. An easy induction shows that if X is Hausdorff and $x_1, \ldots, x_n \in X$, then there are neighborhoods U_1 of x_1, ..., U_n of x_n, such that the U's are pairwise disjoint.

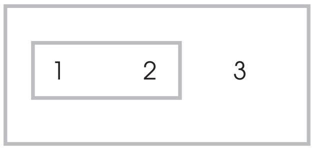
Example. $\mathbb{R}$ is Hausdorff in the standard topology.

Let x and y be distinct points in $\mathbb{R}$. Let $\epsilon = \dfrac{1}{2}|x - y|$. Then $(x - \epsilon, x + \epsilon)$ and $(y - \epsilon, y + \epsilon)$ are disjoint open sets containing x and y, respectively.



□

Example. Consider the topology on $X = \{1, 2, 3\}$ in which the open sets are $\emptyset$, $\{1, 2\}$, and $\{1, 2, 3\}$.



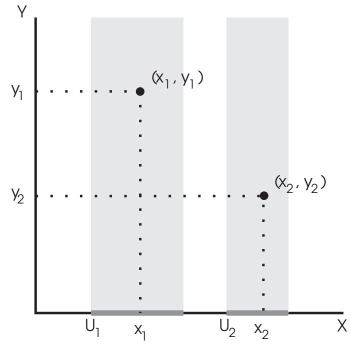
X is not Hausdorff, since it isn't possible to find disjoint neighborhoods containing 1 and 2. □

Proposition. The product of Hausdorff spaces is Hausdorff.

Proof. Let X and Y be Hausdorff spaces. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$, and assume that $(x_1, y_1) \neq (x_2, y_2)$. I want to find disjoint neighborhoods in $X \times Y$ containing the points.

Since the points aren't the same, they must differ in at least one coordinate. Assume $x_1 \neq x_2$ (the argument is similar if instead $y_1 \neq y_2$.

Since X is Hausdorff, I can find disjoint neighborhoods U_1 of x_1 and U_2 of x_2. Then $U_1 \times Y$ and $U_2 \times Y$ are disjoint neighborhoods in $X \times Y$, $(x_1, y_1) \in U_1 \times Y$, and $(x_2, y_2) \in U_2 \times Y$.



Therefore, $X \times Y$ is Hausdorff. □

Proposition. A subspace of a Hausdorff space is Hausdorff.

Proof. Let X be Hausdorff, and let Y be a subspace of X. Let $y_1, y_2 \in Y$, where $y_1 \neq y_2$.

Since X is Hausdorff, there are disjoint neighborhoods U_1 of y_1 and U_2 of y_2 in X. Then $U_1 \cap Y$ is a neighborhood of y_1 in Y and $U_2 \cap Y$ is a neighborhood of y_2 in Y, and $U_1 \cap Y$ and $U_2 \cap Y$ are disjoint.

Therefore, Y is Hausdorff. □

Proposition. If X is Hausdorff and $x_1, \ldots x_n \in X$, then $\{x_1, \ldots, x_n\}$ is closed.

Proof. First, I'll show that a single point is closed. Let $x \in X$. I claim that $X - \{x\}$ is open.

Take $y \in X - \{x\}$. Since $x \neq y$, I may find disjoint neighborhoods U of x and V of y. Since V doesn't intersect U and $x \in U$, it follows that $V \subset X - \{x\}$. Thus, V is a neighborhood of y contained in $X - \{x\}$. Since y was arbitrary, $X - \{x\}$ is open.

Since a finite union of closed sets is closed, if $x_1, \ldots x_n \in X$, then $\{x_1, \ldots, x_n\}$ is closed. □

If y is a limit point of a set Y, then every neighborhood of y meets Y in a point other than y. In a Hausdorff space, every neighborhood of a limit point meets the set in *infinitely many points*.

Proposition. Let X be Hausdorff, and let $Y \subset X$. x is a limit point of Y if and only if every neighborhood of x meets Y in infinitely many points.

Proof. If every neighborhood of x meets Y in infinitely many points, then surely every neighborhood meets Y in a point other than Y. Therefore, x is a limit point of Y.

Conversely, suppose x is a limit point of Y. Let U be a neighborhood of x. I must show that U meets Y in infinitely many points.

Suppose U meets Y in only finitely many points. Then U meets $Y - \{x\}$ in only finitely many points $x_1, \ldots, x_n$. (Note that U must meet $Y - \{x\}$ in *at least one point*, since x is a limit point of Y.)

Now $X - \{x_1, \ldots, x_n\}$ is open and contains x, so $U \cap (X - \{x_1, \ldots, x_n\})$ is open and contains x. But in forming this set I've thrown out the points $x_1, \ldots, x_n$ where U meets Y, so it follows that $U \cap (X - \{x_1, \ldots, x_n\})$ does not meet Y at all! This contradicts the fact that every neighborhood of x must intersect Y.

Hence, U intersects Y in infinitely many points. □