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Homeomorphisms

Definition. Let X and Y be topological spaces, and let $f : X \to Y$. f is a **homeomorphism** if f is bijective, and both f and f^{-1} are continuous.

If there is a homeomorphism between topological spaces X and Y, then X and Y are **homeomorphic**.

Homeomorphic spaces are "the same" as topological spaces, in the same way that isomorphic groups are "the same" as groups. The big problem in topology — which is much too difficult to deal with in this generality — is to classify all spaces up to homeomorphism.

The following properties are obvious:

1. The identity map of a topological space is a homeomorphism.
2. The composite of homeomorphisms is a homeomorphism.
3. The inverse of a homeomorphism is a homeomorphism.

Example. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x + k,$$

where $k \in \mathbb{R}$. Then f is a homeomorphism.

It's clear that f is continuous, since the inverse image of an open interval is an open interval — it's just a translate of the original interval. Then the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(x) = x - k$ is also continuous, since it can be written as $f^{-1}(x) = x + (-k)$.

Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = rx,$$

where $r \in \mathbb{R}$ and $r \neq 0$. Then g is a homeomorphism.

Again, it's clear that g is continuous, since the inverse image of an open interval is an open interval. Replacing r with $\dfrac{1}{r}$ shows that the inverse $g^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous.

From these facts, it follows that any two nonempty open intervals in $\mathbb{R}$ (with the subspace topologies) are homeomorphic. For if (a, b) is a nonempty open interval, then:

1. $f(x) = x - a$ maps (a, b) to $(0, b - a)$.
2. $g(x) = \dfrac{x}{b - a}$ maps $(0, b - a)$ to $(0, 1)$.

f and g are homeomorphisms, by observations above. This proves that every open interval is homeomorphic to $(0, 1)$. But homeomorphism is transitive (since the composite of homeomorphisms is a homeomorphism), so any two nonempty open intervals are homeomorphic.

The function $\tan : \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R}$ is continuous, and its inverse $\arctan : \mathbb{R} \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is continuous. Therefore, $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is homeomorphic to $\mathbb{R}$. But every nonempty open interval is homeomorphic to $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, so it follows that every nonempty open interval is homeomorphic to $\mathbb{R}$. $\square$

Definition. Let $f : X \to Y$ be a function between topological spaces. f is **open** if f takes open sets to open sets. Likewise, f is **closed** if f takes closed sets to closed sets.

Lemma. Homeomorphisms are open maps and closed maps.

Proof. I'll do the open map case; the closed map case is similar.

Let $f : X \to Y$ be a homeomorphism, and let U be open in X. I want to show that $f(U)$ is open in Y.

Since $f : X \to Y$ is a homeomorphism, it has a continuous inverse f^{-1}. By continuity, $(f^{-1})^{-1}(U)$ is open in Y. But $(f^{-1})^{-1}(U) = f(U)$, so $f(U)$ is open in Y. $\square$

Example. You can tell that two spaces are *not* homeomorphic if you can find a topological property that one has that the other does not. In a sense, this is a circular observation: A *topological property* is a property preserved by homeomorphisms. So this observation becomes useful as you develop a repertoire of properties that are preserved by homeomorphisms.

For example, the circle

$$S^1 = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1\right\}$$

with the subspace topology is not homeomorphic to the interval $(0, 1)$ in $\mathbb{R}$ with the subspace topology. One reason: Removing a point from $(0, 1)$ leaves a disjoint union of two sets open in $(0, 1)$, but removing a point from S^1 does not leave a disjoint union of two sets open in S^1.

Take for granted that the assertion about S^1 is true. To show that this difference implies that the spaces aren't homeomorphic, suppose $f : (0, 1) \to S^1$ is a homeomorphism. Take $x \in (0, 1)$, so $(0, 1) - \{x\} = (0, x) \cup (x, 1)$, a disjoint union of open subsets of $(0, 1)$.

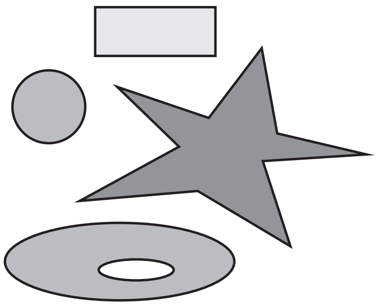
Since f is a homemorphism, $f[(0, x)]$ and $f[(x, 1)]$ are disjoint open subsets of S^1, and $S^1 - \{f(x)\} = f[(0, x)] \cup f[(x, 1)]$. This contradicts my claim about S^1, so S^1 and $(0, 1)$ aren't homeomorphic.

The assertion about S^1 will follow from results I'll prove later about **connected set**. $\square$

Example. Removing a point from $\mathbb{R}$ leaves a disjoint union of two open sets. The same is not true when a point is removed from $\mathbb{R}^n$ for $n \geq 2$. Therefore, $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 2$.

The fact that $\mathbb{R}^m$ and $\mathbb{R}^n$ aren't homeomorphic for $m \neq n$ is a fact called **invariance of domain**; the best proofs use methods from **algebraic topology**. $\square$

The standard informal example of homeomorphic spaces is that a coffee cup is homeomorphic to a doughnut. Popular accounts often say that two spaces are homeomorphic if you can imagine deforming one into the other with cutting, tearing, or breaking.

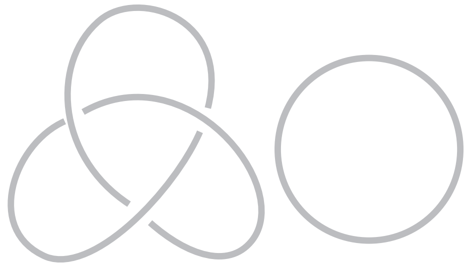


One of these things is not like the others.

In the picture above, the three solid objects are homeomorphic; the object shaped like a ring is not homeomorphic to the others. This kind of assertion (like the one about the coffee cup and the doughnut) must be taken in the right spirit. No one would set out to *prove* the assertion by writing down specific homeomorphisms!

The notion of homeomorphism does not include the idea of "deforming" one thing into another. In the first place, a deformation requires a "big space" inside of which the deformation takes place. In the second place, a deformation is something which takes places in time. Neither of these notions is part of the definition of a homeomorphism. The notion of **isotopy** comes closer to the idea of a continuous deformation of one space into another.

Homeomorphism is also independent of the **embedding** of a space in a larger space. For example, the trefoil knot on the left and the circle on the right are homeomorphic as topological spaces:



In order to get a hold of the obvious difference that exists between the two, you need to consider the *complements* of these sets in $\mathbb{R}^3$. In this case, the ambient space $\mathbb{R}^3$ is responsible for the "knottedness" of the trefoil.