Limit Point Compactness and Sequential Compactness

Definition. Let X be a topological space.

1. X is **limit point compact** if every infinite subset has a limit point.

2. X is **sequentially compact** if every sequence in X has a convergent subsequence.

If X is sequentially compact, then X is limit point compact. For every infinite subset of X contains an infinite sequence; the limit of a convergent subsequence is a limit point of the original subset. A partial converse is contained in the following theorem.

Theorem. Let X be a topological space.

1. If X is compact, then X is limit point compact.

2. If X is a metric space and X is limit point compact, then X is sequentially compact.

Proof. 1. Let X be compact, and let $S \subset X$ be an infinite subset. I must show that S has a limit point.

Suppose that S does not have a limit point. It is vacuously true that S contains all its limit points, so S is closed. Since X is compact, S is compact.

Let $x \in S$. Since x is not a limit point of S, there is a neighborhood U_x of x which does not intersect S in a point other than x. $\{U_x\}_{x \in S}$ is an open cover of S, so by compactness there is a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$.

Now $S \subset \{U_{x_1} \cup \ldots \cup U_{x_n}$, but each U_{x_i} meets S only in x_i. Therefore, $S = \{x_1, \ldots, x_n\}$, which contradicts the assumption that S is infinite.

Therefore, S has a limit point.

2. Suppose X is a metric space and X is limit point compact. Let $\{x_n\}$ be a sequence in X. I want to show that $\{x_n\}$ has a convergent subsequence.

First, I have to eliminate any repetitions in the sequence, so define

$$S = \{x_n \mid n \in \mathbb{Z}^+\}.$$

Thus, S is the *set of points* in $\{x_n\}$.

Suppose S is finite. Then for some $x \in X$, $x_n = x$ for infinitely many $n \in \mathbb{Z}^+$. Suppose this is true for $n_1, n_2, \ldots$. Then $\{x_{n_i}\}_{i \in \mathbb{Z}^+}$ is a convergent subsequence of $\{x_n\}$, because all the terms are equal to x.

The only other possibility is that S is infinite. Then by limit point compactness, S has a limit point x. Consider the nested sequence of neighborhoods $\left\{B\left(x; \dfrac{1}{k}\right) \,\middle|\, k \in \mathbb{Z}^+\right\}$. Since metric spaces are Hausdorff and x is a limit point of S, each $B\left(x; \dfrac{1}{k}\right)$ contains infinitely many points of S.

Choose $n_1 \in \mathbb{Z}^+$ such that $x_{n_1} \in B(x; 1)$. Assuming that $k > 1$ and that n_{k_1} has been chosen, choose $n_k > n_{k-1}$ such that $x_{n_k} \in B\left(x; \dfrac{1}{k}\right)$.

I now have $x_{n_k} \to x$. For if $\epsilon > 0$, choose N so that $\dfrac{1}{n_k} < \epsilon$ for $k \geq N$. Then $x_{n_k} \in B(x; \epsilon)$ for $k \geq N$.
$\square$

Example. The subsets $\{[2n-1, 2n] \mid n \in \mathbb{Z}^+\}$ form a partition of $\mathbb{Z}^+$. The corresponding partition topology is called the **odd-even topology**. Let X denote $\mathbb{Z}^+$ with this topology.

X is limit point compact. To see this, let A be an arbitrary nonempty subset of X, and let $a \in A$. For some $n \in \mathbb{Z}^+$, $a \in [2n-1, 2n]$. Notice that $[2n-1, 2n]$ is the smallest open set containing $2n-1$ or $2n$.

If $a = 2n - 1$, then $2n$ is a limit point of A, since every neighborhood of $2n$ contains a.

If $a = 2n$, then $2n - 1$ is a limit point of A, since every neighborhood of $2n - 1$ contains a.

In either case, A has a limit point. Since A was an arbitrary nonempty subset of X, surely every infinite subset of X has a limit point. Thus, X is limit point compact.

On the other hand, X is not sequentially compact. For example, the sequence $\{1, 2, 3, \ldots\}$ has no convergent subsequence.

This example shows that limit point compactness does not in general imply sequential compactness.

In fact, X is not compact: The open cover $\{[2n-1, 2n] \mid n \in \mathbb{Z}^+\}$ does not have a finite subcover. Thus, limit point compactness does not in general imply compactness. $\square$

Recall that if Y is a subset of a metric space (X, d), the **diameter** of Y is

$$\operatorname{diam}(Y) = \sup\{d(x, y) \mid x, y \in Y\}.$$

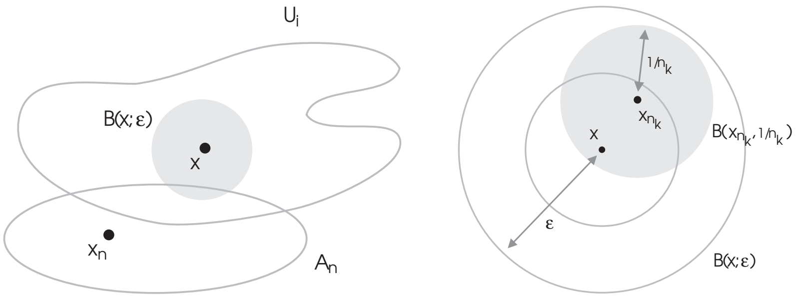
Theorem. (**Lebesgue Number Lemma**) Let X be a sequentially compact metric space, and let $\{U_i\}_{i \in I}$ be an open cover of X. There is an $\epsilon > 0$ such that every subset of X of diameter less than ϵ is contained in an element of $\{U_i\}_{i \in I}$.

ϵ is called a **Lebesgue number** for the cover.

Proof. Suppose on the contrary that no such ϵ exists. Then for every $n \in \mathbb{Z}^+$, there is a set A_n of diameter less than $\dfrac{1}{n}$ which is not contained in any U_i. Let $x_n \in A_n$. I'll show that $\{x_n\}$ has no convergent subsequence, which will contradict sequential compactness.

Suppose on the contrary that $\{x_{n_k}\}$ is a convergent subsequence converging to x. Suppose that $x \in U_i$. Find $\epsilon > 0$ such that $B(x; \epsilon) \subset U_i$. Then choose n_k sufficiently large so that

$$d(x, x_{n_k}) < \frac{\epsilon}{2} \quad \text{and} \quad \frac{1}{n_k} < \frac{\epsilon}{2}.$$



Since $\operatorname{diam}(A_{n_k}) < \dfrac{1}{n_k}$ and $x_{n_k} \in A_{n_k}$, it follows that $A_{n_k} \subset B\left(x_{n_k}; \dfrac{1}{n_k}\right)$. Suppose $y \in B\left(x_{n_k}; \dfrac{1}{n_k}\right)$. Then

$$d(x, y) \le d(x, x_{n_k}) + d(x_{n_k}, y) \le \frac{\epsilon}{2} + \frac{1}{n_k} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $B\left(x_{n_k}; \dfrac{1}{n_k}\right) \subset B(x; \epsilon)$. Hence,

$$A_{n_k} \subset B\left(x_{n_k}; \frac{1}{n_k}\right) \subset B(x; \epsilon) \subset U_i.$$

This contradicts the fact that A_{n_k} is not contained in any U_i. Therefore, there *is* a number ϵ satisfying the conclusion of the theorem. $\square$

Theorem. Let X be a metric space. The following are equivalent:

1. X is compact.
2. X is limit point compact.
3. X is sequentially compact.

Proof. I've already proven ($1 \Rightarrow 2$) and ($2 \Rightarrow 3$). I need to prove ($3 \Rightarrow 1$). Suppose then that X is sequentially compact. I need to show that X is compact.

Step 1. For every $\epsilon > 0$, there is a finite covering of X by ϵ-balls.

Let $x_1 \in X$ and construct $B(x_1;\epsilon)$. If $X = B(x_1;\epsilon)$, then I'm done. Otherwise, choose $x_2 \in X - B(x_1;\epsilon)$ and construct $B(x_2;\epsilon)$. Again, if $X = B(x_1;\epsilon) \cup B(x_2;\epsilon)$, I'm done; otherwise, choose $x_3 \in X - (B(x_1;\epsilon) \cup B(x_2;\epsilon))$. Keep going. Notice that each $B(x_n;\epsilon)$ contains only one x_k, namely x_n.

Suppose that the process does not terminate. Consider the sequence $\{x_n\}$. I claim that it has no convergent subsequence. Indeed, if $\{x_{n_k}\}$ is a subsequence converging to x, then $B\left(x;\frac{\epsilon}{2}\right)$ can contain at most one x_{n_k}: If it contains two x_{n_k}'s, then they are less than ϵ apart. Therefore, there is an ϵ-ball about an x_{n_k} containing another x_{n_k}

It follows that the process *must* terminate, so X is covered by a finite number of ϵ-balls.

Step 2. Every open cover of X contains a finite subcover.

Let $\{U_i\}_{i\in I}$ be an open cover of X. Let ϵ be a Lebesgue number for the cover. Cover X with a finite number of $\frac{\epsilon}{2}$ balls $\{B_1,\ldots,B_n\}$. Each B_j has diameter less than ϵ, so each B_j is contained in some U_{i_j}. Then $\{U_{i_j} \mid j = 1,\ldots,n\}$ is a finite subcollection of $\{U_i\}_{i\in I}$ which covers X. Therefore, X is compact. □

Definition. Let X and Y be metric spaces, and let $f : X \to Y$. f is **uniformly continuous** if for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_X(a,b) < \delta$ implies $d_Y(f(a),f(b)) < \epsilon$ for all $a,b \in X$.

Theorem. Let X and Y be metric spaces, let $f : X \to Y$ be a continuous function, and suppose X is compact. Then f is uniformly continuous.

Proof. Given $\epsilon > 0$, the open balls $\left\{B\left(y;\frac{\epsilon}{2}\right) \;\middle|\; y \in Y\right\}$ cover Y. Therefore, $\left\{f^{-1}\left(B\left(y;\frac{\epsilon}{2}\right)\right) \;\middle|\; y \in Y\right\}$ is an open cover of X.

By the Lebesgue Number Lemma, there is a $\delta > 0$ such that every set of diameter less than δ is contained in a $f^{-1}\left(B\left(y;\frac{\epsilon}{2}\right)\right)$. In particular, if $d_X(a,b) < \delta$, then $\{a,b\}$ is a set of diameter less than δ, so $a,b \in f^{-1}\left(B\left(y;\frac{\epsilon}{2}\right)\right)$ for some y. This means that $f(a), f(b) \in B\left(y;\frac{\epsilon}{2}\right)$, so $d_Y(f(a),f(b)) < \epsilon$. □

Example. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. f is continuous, but f is not uniformly continuous.

To show that f is not uniformly continuous, I'll show that there is no $\delta > 0$ such that if $\delta > |x - y|$, then $1 > |f(x) - f(y)|$. Suppose on the contrary that such a δ exists. Choose $x > \frac{2}{\delta}$, and consider the points $x + \frac{\delta}{2}$ and x.

First,

$$\left(x + \frac{\delta}{2}\right) - x = \frac{\delta}{2} < \delta.$$

The points are less than δ units apart.

However,

$$\left(x + \frac{\delta}{2}\right) + x = 2x + \frac{\delta}{2} > 2\cdot\frac{2}{\delta} + \frac{\delta}{2} = \frac{4}{\delta} + \frac{\delta}{2} > \frac{4}{\delta}.$$

So

$$f\left(x+\frac{\delta}{2}\right)-f(x)=\left(x+\frac{\delta}{2}\right)^{2}-x^{2}=\left(x+\frac{\delta}{2}-x\right)\left(x+\frac{\delta}{2}+x\right)=\frac{\delta}{2}\cdot\left(x+\frac{\delta}{2}+x\right)>\frac{\delta}{2}\cdot\frac{4}{\delta}=2.$$

The images are more than 1 unit apart.
This shows that f is not uniformly continuous. $\square$
