Local Compactness

Definition. Let X be a topological space. X is **locally compact** if for all $x \in X$, there is a compact set C and a neighborhood U of x such that $x \in U \subset C$.

If X is compact, then X is locally compact: For any $x \in X$, X is a compact set containing the neighborhood X of x.

Example. If X is a space with the discrete topology, then X is locally compact. If $x \in X$, then $\{x\}$ is a compact set containing the neighborhood $\{x\}$ of x. $\square$

Example. $\mathbb{R}^n$ is locally compact: If $x \in \mathbb{R}^n$, the closed ball $\overline{B(x;1)}$ is compact, and it contains the open ball $B(x;1)$. $\square$

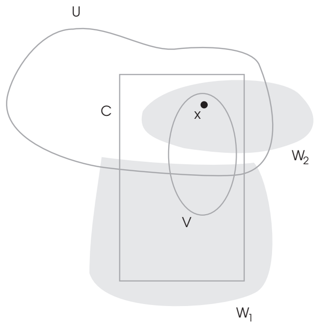
Theorem. Let X be a Hausdorff space. X is locally compact if and only if for every $x \in X$ and every neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Proof. Suppose for every $x \in X$ and every neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$. Choose any neighborhood U of x — for example, X — and find a neighborhood V of x with compact closure such that $\overline{V} \subset U$. Then $\overline{V}$ is a compact set containing a neighborhood V of x, so X is locally compact.

Conversely, suppose X is locally compact. Let $x \in X$, and let U be a neighborhood of x. Let C be a compact set which contains a neighborhood V of x.

C is compact and X is Hausdorff, therefore C is closed. It follows that $C - U = C \cap (X - U)$ is closed; since it's a subset of the compact set C, $C - U$ is compact. Note that $x \in U$, so $x \notin C - U$.

I showed earlier that in a Hausdorff space, a compact set and a point not contained in it may be separated by disjoint open sets. Thus, I may find disjoint open sets W_1 and W_2 such that $C - U \subset W_1$ and $x \in W_2$.



Next, I'll show that $\overline{W_2 \cap V}$ is compact. C is a closed set containing V, so $\overline{V} \subset C$. But $W_2 \cap V \subset V$, so $\overline{W_2 \cap V} \subset \overline{V} \subset C$. Now $\overline{W_2 \cap V}$ is a closed subset of the compact set C, so $\overline{W_2 \cap V}$ is compact.

Moreover, $W_2 \cap V \subset W_2 \subset X - W_1$. $X - W_1$ is closed, so $\overline{W_2 \cap V} \subset X - W_1$. Since $\overline{W_2 \cap V}$ does not meet W_1, and since $C - U \subset W_1$, it follows that $\overline{W_2 \cap V}$ does not intersect $C - U$. But $\overline{W_2 \cap V}$ *is* a subset of C, so $\overline{W_2 \cap V} \subset U$.

Thus, $W_2 \cap V$ is a neighborhood of x with compact closure whose closure is contained in U. $\square$

Corollary. Let X be a locally compact Hausdorff space, and let Y be a subspace of X. If Y is open or closed in X, then Y is locally compact.

Proof. Suppose Y is open in X. Let $x \in Y$. By the preceding result, I may find a neighborhood U of x such that $\overline{U}$ is compact and $\overline{U} \subset Y$. Note that U is automatically open in Y. Thus, x is contained in a compact set which contains a Y-neighborhood of x. Therefore, Y is locally compact.

Suppose Y is closed. Let $y \in Y$, and let C be a compact set containing a neighborhood U of y. C is closed, since X is Hausdorff; therefore, $C \cap Y$ is a closed subset of C. Since C is compact, $C \cap Y$ is compact. Moreover, $C \cap Y$ contains $U \cap Y$, which is a neighborhood of y. Therefore, Y is locally compact. $\square$

Example. Let S denote the topologist's sine curve

$$S = \left\{ \left(x, \sin \frac{1}{x} \right) \,\Big|\, 0 < x \le 1 \right\}.$$

Let $X = S \cup \{(0, 0)\}$. X is not locally compact.

Take a ball $B((0, 0); \epsilon)$ about the origin. Consider the intersection of X with the line $y = \dfrac{\epsilon}{2}$. The intersection is an infinite sequence of points with all but finitely many terms lying inside $B((0, 0); \epsilon)$. The sequence converges to $\left(0, \dfrac{\epsilon}{2}\right)$, which is not in X.

If there is a compact set C in X containing a neighborhood of $(0, 0)$, then that neighborhood contains $B((0, 0); \epsilon) \cap X$ for ϵ sufficiently small. By the argument above, such a neighborhood contains an infinite subset of X with no limit point. The same is true for C, which contradicts the fact that compact sets are limit point compact. $\square$

It is often useful to embed a space in a compact Hausdorff space, because compact Hausdorff spaces are very nicely behaved. For example, I'll show later that compact Hausdorff spaces are **normal**: Any two disjoint closed sets can be separated by disjoint open sets. If a space is locally compact Hausdorff, it can be embedded in a compact Hausdorff space called the **one-point compactification**.

Lemma. Let X be a locally compact Hausdorff space. Define $Y = X \cup \{\infty\}$, where ∞ is a point not in X. Let $\mathcal{T}$ be the collection of subsets of Y consisting of:

1. Any open subset of X.
2. The complement in Y of a compact subset of X.

Then $\mathcal{T}$ is a topology on Y.

$(Y, \mathcal{T})$ is called the **one-point compactification** of X.

Proof. $\emptyset$ is open in X, so it's open in Y. $\emptyset$ is compact, so $Y = Y - \emptyset$ is open in Y.

To verify the axioms for unions and intersections, it's necessary to take cases. I'll show the work for unions; the proof for intersections is similar.

The union of open sets in X is open in X, so it's open in Y.

If $\{C_i\}$ is a family of compact subsets of X, then

$$\bigcup_{i \in I} (Y - C_i) = Y - \bigcap_{i \in I} C_i.$$

Now X is Hausdorff, so compact subsets are closed. Therefore, $\bigcap_{i \in I} C_i$ is closed. Since it's contained in any one of the compact sets C_i, it's also compact. Therefore, $\bigcup_{i \in I} (Y - C_i)$ is the complement in Y of a compact set in X, so it's open in Y.

The remaining possibility is a union of sets open in X with complements of compact subsets of X. Using the first two cases, this reduces to showing that if U is open in X and C is a compact subset of X, then $U \cup (Y - C)$ is open in Y. But

$$U \cup (Y - C) = Y - (C - U).$$

C is closed in X, U is open in X, so $C - U$ is closed in X. But $C - U \subset C$ and C is compact, therefore $C - U$ is compact. Hence, $U \cup (Y - C)$ is a complement in Y of a compact set in X, so it's open in Y. $\square$

Here are some properties of the one-point compactification.

Lemma. Let X be a locally compact Hausdorff space, and let Y be its one-point compactification.

1. The subspace topology on X is the same as the original topology on X. (That is, the inclusion of X into Y is a homeomorphism onto its image.)

2. If X is not compact, then $Y = \overline{X}$.

3. Y is compact.

4. Y is Hausdorff.

Proof. 1. If U is open in X, then it's open in Y by definition. Conversely, an open set in Y is either an open set U in X (in which case $U \cap X = U$ is open in X) or $Y - C$, where C is compact in X. But $(Y - C) \cap X = X - C$, which is open in$X$ since C is closed in X.

2. Since the only point of Y not in X is ∞, this amounts to showing that ∞ is a limit point of X. Neighborhoods of ∞ are sets $Y - C$, where C is compact, and since X is not compact, $C \neq X$. Thus, $Y - C$ must intersect X, and so $\infty \in \overline{X}$.

3. Let $\{U_i\}_{i \in I}$ be an open cover of Y. One of the U's, say U_0, must contain ∞, so U_0 must be a set of the form $Y - C$, where C is compact in X. The sets $U_i \cap X$ for $U_i \neq U_0$ form an open cover of C; let $\{U_{i_1} \cap X, \ldots, U_{i_n} \cap X\}$ be a finite subcover. Then $\{U_{i_1}, \ldots, U_{i_n}, U_0\}$ covers Y.

4. Let x and y be distinct points of Y. If they're both in X, they may be separated by disjoint neighborhoods in X (since X is Hausdorff), and these neighborhoods are also open in Y.

Otherwise, I'm trying to separate $x \in X$ from ∞. Since X is locally compact, I may find a compact set C containing a neighborhood U of x. Then $Y - C$ and U are disjoint open sets in Y separating x and ∞. $\square$

Corollary. A space X embeds as an open subset of a compact Hausdorff space if and only if X is locally compact Hausdorff. $\square$