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Path-Connectedness and Local Connectedness

Definition. Let X be a topological space. A **path** in X is a continuous function $\sigma : [0,1] \to X$. $\sigma(0)$ and $\sigma(1)$ are the **endpoints** of the path; σ is **a path from** $\sigma(0)$ **to** $\sigma(1)$.

Definition. Let X be a topological space. X is **path connected** if for all $x, y \in X$, there is a path in X from x to y.

Example. Points are path connected. $\square$

Example. $\mathbb{R}^n$ is path connected. If $x, y \in \mathbb{R}^n$, then

$$\sigma(t) = (1-t)x + ty$$

is a path in $\mathbb{R}^n$ from x to y. $\square$

Lemma. A path connected space is connected.

Proof. Suppose X is path connected, and suppose that $X = U \cup V$ is a separation of X. Let $x \in U$ and let $y \in V$. By assumption there is a path $\sigma : [0,1] \to X$ such that $\sigma(0) = x$ and $\sigma(1) = y$.

$\sigma^{-1}(U)$ and $\sigma^{-1}(V)$ are disjoint open sets in $[0,1]$, and $[0,1] = \sigma^{-1}(U) \cup \sigma^{-1}(V)$. This contradicts the fact that $[0,1]$ is connected.

Therefore, X is connected. $\square$

Example. The **topologist's sine curve** S consists of the graph of $y = \sin \dfrac{1}{x}$ for $0 < x \le 1$.

The topologist's sine curve is connected, since it's a continuous image of the connected set $(0,1]$. The space $T = S \cup \{(0,0)\}$ is also connected, since it's the union of S with one of its limit points in $\mathbb{R}^2$. However, T is not path connected.

To prove this, I'll show that the point $(0,0)$ cannot be connected by a path to the point $(1, \sin 1)$.

Suppose on the contrary that $\sigma : [0,1] \to \overline{S}$ is a path from $(0,0)$ to $(1, \sin 1)$. Notice that $(0,0)$ is closed in T, so $\sigma^{-1}((0,0))$ is a closed subset of $[0,1]$. I will show that $\sigma^{-1}((0,0))$ is also open in $[0,1]$; this will prove that $\sigma^{-1}((0,0)) = [0,1]$.

Let $x \in \sigma^{-1}((0,0))$. Let V be the open rectangle $(-0, 5.0.5) \times (-0.5, 0.5)$. The intersection of V with $\overline{S}$ is an open set in $\overline{S}$ containing $(0,0)$ — that is, a neighborhood of $(0,0)$ in $\overline{S}$. By continuity, I may find an open interval (a,b) in $\mathbb{R}$ containing x such that

$$\sigma\left([0,1] \cap (a,b)\right) \subset V \cap \overline{S}.$$

The intersection $[0,1] \cap (a,b)$ is an open interval or a half-open interval, so it's connected. Therefore, $\sigma\left([0,1] \cap (a,b)\right)$ is connected. $V \cap \overline{S}$ is a disjoint union of open arcs in S with the point $(0,0)$. Hence, $\sigma\left([0,1] \cap (a,b)\right)$ must map entirely into one of these components.

However, I know the point $x \in [0,1] \cap (a,b)$ maps to $(0,0)$. Therefore,

$$\sigma\left([0,1] \cap (a,b)\right) = (0,0).$$

Thus, $[0,1] \cap (a,b)$ is a neighborhood of x contained in $\sigma^{-1}((0,0))$. This proves that $\sigma^{-1}((0,0))$ is open.

Since $\sigma^{-1}((0,0))$ is a nonempty, open, and closed, and since it's a subset of the connected set $[0,1]$, it must be all of $[0,1]$. Thus, $\sigma([0,1]) = (0,0)$, contradicting the fact that σ is a path from $(0,0)$ to $(1, \sin 1)$.

It follows that there is no such path, and therefore T isn't path connected. $\square$

Connected components were defined as the maximal connected subsets of a space; their existence followed from Zorn's Lemma. The **path components** of a space are defined in a similar fashion. I need a lemma which is analogous to one I prove for connected sets.

Lemma. Let $\{U_a\}_{a \in A}$ be a collection of path connected subsets of a topological space X. Suppose that $\bigcap_{a \in A} U_a \neq \emptyset$. Then $U = \bigcup_{a \in A} U_a$ is path connected.

Proof. Let $y, z \in U$. I must show that y and z can be joined by a path in U.

Suppose $y \in U_a$ and $z \in U_b$. Since $x \in U_a$ and U_a is path connected, x and y can be joined by a path in U_a. Since $x \in U_b$ and U_b is path connected, x and z can be joined by a path in U_b.

Concatenating the two paths produces a path in $U_a \cup U_b \subset U$ joining y and z. Therefore, U is path connected. $\square$

Theorem. Let X be a topological space.

1. Every point $x \in X$ is contained in a unique maximal path connected subset C_x of X.
2. $C_x = C_y$ or $C_x \cap C_y = \emptyset$. $\square$

To say that C_x is maximal means that C_x is not properly contained in any other path connected subset of X.

The proof will be omitted; it is essentially the same as the result for connected components. You can verify that the connected components proof required only the lemma on unions of connected sets with nonempty intersection; I just proved the analogous lemma for path connected sets above.

Definition. Let X be a topological space. The **path components** of X are the maximal path connected subsets of X.

Definition. Let X be a topological space.

1. X is **locally connected** if for every $x \in X$ and every neighborhood U of x, there is a connected neighborhood V of x such that $V \subset U$.
2. X is **locally path connected** if for every $x \in X$ and every neighborhood U of x, there is a path connected neighborhood V of x such that $V \subset U$.

Example. Since a path connected neighborhood is a connected neighborhood, every locally path connected space is locally connected. $\square$

Example. $\mathbb{R}^n$ is locally path connected, since a ball $B(x;\epsilon)$ is path connected. $\square$

Example. Let S denote the topologist's sine curve:

$$S = \left\{ \left(x, \sin\frac{1}{x}\right) \;\middle|\; 0 < x \leq 1 \right\}.$$

Let $T = S \cup \{(0,0)\}$. T is not locally connected; a neighborhood of $(0,0)$ consists of $(0,0)$ together with a disjoint union of open arcs, which is not a connected set. $\square$

Theorem. A space is locally connected if and only if the components of any open subset are open.

In particular, the components of a locally connected space are open.

Proof. Suppose X is connected and $U \subset X$ is open. Let C be a component of U. I want to show C is open.

Let $x \in C$. By local connectedness, there is a connected neighborhood V of x such that $x \in V \subset U$. Now $V \cup C$ is a union of connected sets having the point x in common; since C is maximal, $V \cup C = C$, so $V \subset C$.

Since every point of C has a neighborhood contained in C, C is open.

Conversely, suppose that components of open subsets are open. Let $x \in X$, and let U be a neighborhood of x. I must find a connected neighborhood V of x such that $x \in V \subset U$. Take V to be the connected component of U which contains x. By assumption, V is open, and $x \in V \subset U$.

Hence, X is locally connected. □

An essentially identical argument proves the following.

Theorem. A space is locally path connected if and only if the path components of any open set are open. □

In particular, the path components of a locally path connected space are open.

The relationship between components and path components is described by the following result.

Proposition. Let X be a topological space.

1. Every path component is a subset of a component.

2. If X is locally path connected, then the path components and components coincide.

Proof. 1. A path component is path connected, and path connected sets are connected. Since every connected set is contained in a component (i.e. a maximal connected set), every path component is contained in a component.

2. Suppose X is locally path connected. Let C be a component. I have to show that C is a path component.

Let P be any path component contained in C. (For instance, let $x \in C$, and take P to be the path component containing x.) Let P' be the union of the path components of C other than P.

Since X is locally path connected, P is open. X is locally connected, so C is open; again, since X is locally path connected, the path components of the open set C are open. Hence, P' is open, since it's a union of open sets.

Now $C = P \cup P'$ is a disjoint union of open sets. Since C is connected, P and P' can't be nonempty; since $P \neq \emptyset$, I have $P' = \emptyset$. Thus, C has no path components other than P, which means that $C = P$. □