6-23-1999

# The Product Topology

If $X$ and $Y$ are topological spaces, the **product topology** on $X \times Y$ is the topology generated by the basis consisting of all sets $U \times V$, where $U$ is open in $X$ and $V$ is open in $Y$. This may be extended to a product of finitely many spaces in the obvious fashion. How do you define a topology on a product of infinitely many spaces?

**Definition.** Let $\{X_a\}_{a\in A}$ be a family of topological spaces. The **box topology** on $\prod_{a\in A} X_a$ is the topology generated by the basis consisting of sets $\prod_{a\in A} U_a$, where $U_a$ is open in $X_a$.

Let $\mathcal{B}$ be the collection of sets $\prod_{a\in A} U_a$, where $U_a$ is open in $X_a$. I want to show that $\mathcal{B}$ is a basis.

Let $(x_a)_{a\in A} \in \prod_{a\in A} X_a$. Since $X_a$ is open in $X_a$, $\prod_{a\in A} X_a$ is an element of $\mathcal{B}$, and it obviously contains $(x_a)_{a\in A}$.

Let $(x_a)_{a\in A} \in \left(\prod_{a\in A} U_a \cap \prod_{a\in A} V_a\right)$, where $U_a$ and $V_a$ are open in $X_a$ for each $a$. Then $U_a \times V_a$ is open in $X_a$, so $\prod_{a\in A}(U_a \cap V_a)$ is an element of $\mathcal{B}$. Moreover,

$$(x_a)_{a\in A} \in \prod_{a\in A}(U_a \cap V_a) \subset \left(\prod_{a\in A} U_a \cap \prod_{a\in A} V_a\right).$$

This proves that $\mathcal{B}$ is a basis.

There is another way to generate a topology on a product. I'll need some preliminary results.

**Lemma.** If $\{\mathcal{T}_a\}_{a\in A}$ is a family of topologies on a set $X$, then $\mathcal{T} = \bigcap_{a\in A} \mathcal{T}_a$ is a topology on $X$.

**Proof.** Since $\emptyset, X \in \mathcal{T}_a$ for all $a$, it follows that $\emptyset, X \in \mathcal{T}$.

If $\{U_i\}_{i\in I}$ is a collection of open sets in $\mathcal{T}$, then for each $a$, I have $U_i \in \mathcal{T}_a$ for all $i$. Since $\mathcal{T}_a$ is a topology, $\bigcup_{i\in I} U_i \in \mathcal{T}_a$ for all $a$. Hence, $\bigcup_{i\in I} U_i \in \mathcal{T}$. Thus, $\mathcal{T}$ is closed under arbitrary unions.

If $\{U_1, \ldots, U_n\}$ is a collection of open sets in $\mathcal{T}$, then for each $a$, I have $U_i \in \mathcal{T}_a$ for all $i = 1, \ldots, n$. Since $\mathcal{T}_a$ is a topology, $\bigcap_{i=1}^n U_i \in \mathcal{T}_a$ for all $a$. Hence, $\bigcap_{i=1}^n U_i \in \mathcal{T}$. Thus, $\mathcal{T}$ is closed under finite intersections.

Therefore, $\mathcal{T}$ is a topology. $\square$

**Definition.** Let $\{X_a\}_{a\in A}$ be a family of topological spaces, let $X$ be a set, and let $\{f_a : X \to X_a \mid a \in A\}$ be a family of functions. The **topology induced on $X$ by the family** $\{f_a\}_{a\in A}$ is the smallest topology on $X$ which makes all the $f_a$'s continuous.

There is at least one topology on $X$ which makes all the $f_a$'s continuous, namely the discrete topology. If I intersect *all* the topologies on $X$ which make the $f_a$'s continuous, I get a topology (by the last lemma) which makes all the $f_a$'s continuous. It is clearly the smallest such topology, in the sense that it's contained in every topology which makes all the $f_a$'s continuous.

While this construction accomplishes the goal of making the $f_a$'s continuous, it would be nice to have a more concrete description of the open sets.

**Lemma.** Let $\{X_a\}_{a\in A}$ be a family of topological spaces, let $X$ be a set, and let $\{f_a : X \to X_a \mid a \in A\}$ be a family of functions. The topology induced on $X$ by the $f_a$'s is the same as the topology generated by the subbasis

$$\left\{f_a^{-1}(U) \mid U \overset{\text{open}}{\subset} X_a, \quad a \in A\right\}.$$

**Proof.** First, this collection is a subbasis: If $x \in X$, fix $a \in A$ and consider $f_a(x) \in X_a$. Then $X_a$ is open in $X_a$ and $f_a^{-1}(X_a)$ is an element of the collection which contains $x$.

Now let $\mathcal{S}$ be the topology induced on $X$ by the $f_a$'s and let $\mathcal{T}$ be the topology generated by the subbasis.

If $U$ is open in $X_a$ and $f_a^{-1}(U)$ is an element of the subbasis, then $f_a^{-1}(U)$ is open in $\mathcal{T}$. Since $U$ and $f_a$ were arbitrary, this shows that $\mathcal{T}$ makes all the $f_a$'s continuous. But $\mathcal{S}$ is the smallest topology which makes the $f_a$'s continuous. Hence, $\mathcal{S} \subset \mathcal{T}$.

Conversely, let $U$ be open in $X_a$ and let $f_a^{-1}(U)$ be an element of the subbasis. I claim that $f_a^{-1}(U)$ is an open set in $\mathcal{S}$. For if not, then $f_a$ is *not* continuous, since the inverse image under $f_a$ of the open set $U$ isn't open in $X$.

Now I know that all the elements of the subbasis are contained in $\mathcal{S}$. But $\mathcal{S}$ is a topology, so arbitrary unions of finite intersections of elements of the subbasis are also contained in $\mathcal{S}$. Therefore, $\mathcal{T} \subset \mathcal{S}$.

Hence, the two topologies are the same. $\square$

Here's an important property of this topology.

**Proposition.** Let $\{X_a\}_{a \in A}$ be a family of topological spaces, let $X$ be a set, and let $\{f_a : X \to X_a \mid a \in A\}$ be a family of functions. Give $X$ the topology induced on $X$ by the $f_a$'s, and let $W$ be a topological space. A function $g : W \to X$ is continuous if and only if $f_a \cdot g : W \to X_a$ is continuous for all $a$.

**Proof.** If $g$ is continuous, then $f_a \cdot g : W \to X_a$ is continuous for all $a$, since $f_a \cdot g$ is a composite of continuous maps.

Conversely, suppose $f_a \cdot g : W \to X_a$ is continuous for all $a$. I want to show that $g$ is continuous. Let $U$ be open in $X$; I must show that $g^{-1}(U)$ is open in $W$.

To begin with, take $U$ to be a subbasic open set. Thus, suppose $U = f_a^{-1}(V)$, where $V$ is open in $X_a$. Then

$$g^{-1}(U) = g^{-1}(f_a^{-1}(V)) = (f_a \cdot g)^{-1}(V).$$

Since $f_a \cdot g$ is continuous, $(f_a \cdot g)^{-1}(V)$ is open in $W$.

For a general open set $U$, simply write $U$ as a union of a finite intersections of subbasic open sets, then use the fact that $g^{-1}(-)$ commutes with unions and intersections. $\square$

Now specialize to the case where $X = \prod_{a \in A} X_a$, and give $X$ the topology induced by the family of projection maps

$$\left\{\pi_b : \prod_{a \in A} X_a \to X_b \mid b \in A\right\}.$$

The topology generated in this way is called the **product topology**; *it is the smallest topology which makes all the projection maps continuous.*

Notice that I've already used the name "product topology" in the case of a product of finitely many spaces to mean the topology generated by the basis consisting of products of open sets from the factors. In fact, if there are finitely many factors, these two topologies are the same.

To see this, it's helpful to get an explicit description for the *basic* open sets in the product topology. A typical basis element has the form

$$\pi_{a_1}^{-1}(U_1) \cap \pi_{a_2}^{-1}(U_2) \cap \ldots \cap \pi_{a_n}^{-1}(U_n),$$

where $U_i$ is open in $X_{a_i}$. Note that $\pi_a^{-1}(U) \cap \pi_a^{-1}(V) = \pi_a^{-1}(U \cap V)$, so I can combine terms with the same index. Thus, I may as well assume that $a_1, a_2, \ldots, a_n$ are distinct.

In this case, the set above is

$$\prod_{a \in A} V_a, \quad \text{where} \quad V_a = \begin{cases} X_a & \text{if } a \neq a_1, a_2, \ldots, a_n \\ U_i & \text{if } a = a_i, \quad i = 1, 2, \ldots, n \end{cases}.$$

*Thus, a typical basis element for the product topology is a product of sets open in the factors, where all but finitely many of these sets are the whole spaces.* In the case where there are only finitely many factors, this is simply products of sets open in the factors – the same as the product topology I defined earlier.

The next result says that a function into a product is continuous if and only if its factors are continuous.

**Corollary.** Let $\{X_a\}_{a \in A}$ be a family of topological spaces, and give $\prod_{a \in A} X_a$ the product topology. Let $W$ be a topological space, and let $g : W \to \prod_{a \in A} X_a$ be a function. $g$ is continuous if and only if $\pi_a \cdot g$ is continuous for all $a$. $\square$

**Example.** Define $f : \mathbb{R} \to \mathbb{R}^3$ by

$$f(t) = (\cos t, \sin t, t).$$

In this case, the projections are

$$(\pi_1 \cdot f)(t) = \cos t, \quad (\pi_2 \cdot f)(t) = \sin t, \quad (\pi_3 \cdot f)(t) = t.$$

Since these are continuous as functions $\mathbb{R} \to \mathbb{R}$, it follows from the corollary that $f$ is continuous as well.

$\square$

A final note: A basis element in the box topology is a product of sets open in the factors. Since this is *more general* than the condition for basis elements of the product topology, it follows that *the box topology is finer than the product topology.*