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Quotient Spaces

You may have encountered **quotient groups** or **quotient rings** in an algebra course. In those situations, the idea was to turn a set of cosets of a subgroup or an ideal into a group or a ring.

A set of cosets is a **partition** of a group or a ring. In the topological setting, a **quotient space** arises as a topology on a set of partition elements. You can use this to construct new topological spaces by "gluing" and "pasting".

If X and Y are topological spaces, a function $f : X \to Y$ is continuous if and only if when V is open in Y, f^{-1} is open in X.

Definition. Let $f : X \to Y$, where X and Y are topological spaces. f is a **quotient map** if:

1. f is surjective.
2. V is open in Y if and only if $f^{-1}(V)$ is open in X.

Remarks.

1. Quotient maps are continuous.
2. If $f : X \to Y$ is a homeomorphism, then f is a quotient map.

   Since f is a homeomorphism, it's continuous; hence, if V is open in Y, then $f^{-1}(V)$ is open in X. Conversely, suppose $V \subset Y$ and $f^{-1}(V)$ is open in X. Since f^{-1} is continuous,

$$V = f\left(f^{-1}(V)\right) = \left(f^{-1}\right)^{-1}\left(f^{-1}(V)\right) \quad \text{is open.}$$

3. The composite of two quotient maps is a quotient map.

In the definition of a quotient map, there are topologies on both X and Y. More often, you start with a topological space and a partition of the space, then try to topologize the set of partition elements so that the associated function (carrying an element of the space to the partition element containing it) is a quotient map.

Lemma. Let X be a topological space, let Y be a set, and let $f : X \to Y$ be a surjection. Define a collection of subsets of Y by requiring that $V \in \mathcal{T}$ if and only if $f^{-1}(V)$ is open in X. Then $\mathcal{T}$ is a topology on Y.

$\mathcal{T}$ is called the **quotient topology** on Y.

Proof. $f^{-1}(\emptyset) = \emptyset$ is open in X, so $\emptyset$ is open in Y. $f^{-1}(Y) = X$ is open in X, so Y is open in Y.

Suppose $\{V_i\}_{i \in I}$ is a collection of subsets of Y such that $f^{-1}(V_i)$ is open in X for all $i \in I$. Now

$$f^{-1}\left(\bigcup_{i \in I} V_i\right) = \bigcup_{i \in I} f^{-1}(V_i),$$

and $\bigcup_{i \in I} f^{-1}(V_i)$ is open in X. Therefore, $\bigcup_{i \in I} V_i$ is open in Y.

Finally, suppose $V_1, \ldots V_n$ is a collection of subsets of Y such that $f^{-1}(V_i)$ is open in X for $i = 1, \ldots, n$.

$$f^{-1}\left(\bigcap_{i=1}^{n} V_i\right) = \bigcap_{i=1}^{n} f^{-1}(V_i),$$

and $\bigcap_{i=1}^{n} f^{-1}(V_i)$ is open in X. Therefore, $\bigcap_{i=1}^{n} V_i$ is open in Y.

Therefore, $\mathcal{T}$ is a topology. $\square$

Example. (**Partition Topologies**) Let X be a topological space, and let $X^* = \{X_a\}_{a \in A}$ be a partition of X. Define $f : X \to X^*$ by

$$f(x) = X_a \quad \text{if} \quad x \in X_a.$$

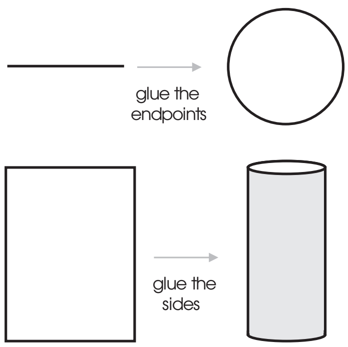
This makes sense, because every element of X lies in exactly one X_a.

f is clearly surjective; the quotient topology on X^* induced by f is called the **partition topology**.

In fact, all quotient topologies are partition topologies. For if X is a topological space and $f : X \to Y$ is surjective, the collection of inverse images $\{f^{-1}(y) \mid y \in Y\}$ forms a partition of X. The partition topology induced by this partition is the same as the quotient topology induced by f.

In applications, you often want to "glue" parts of a space together to make a new space. In this case, the partition elements are as follow. A point that isn't being glued to anything else is in a partition element by itself. Otherwise, a group of points that are being glued together form a partition element.

For example, take the interval $[0, 1]$. Partition $[0, 1]$ by defining $\{0, 1\}$ to be one partition element and defining $\{x\}$ to be a partition element for $0 < x < 1$. This has the effect of gluing the endpoints together; the quotient space is homeomorphic to a circle.



Similarly, if you take a rectangle and glue two opposite sides together, you get a cylinder. In this case, the partition elements consist of individual points not on the glued sides and pairs of corresponding points on the sides being glued.

If you twist the rectangle before gluing the opposite sides, the resulting quotient space is a Möbius strip. If you glue both pairs of opposite sides together with no twisting, you get a torus. $\square$

Lemma. Let X and Y be topological spaces. If $f : X \to Y$ is continuous, open, and surjective, then f is a quotient map.

Proof. If V is open in Y, then $f^{-1}(V)$ is open in X by continuity.

Suppose $V \subset Y$ and $f^{-1}(V)$ is open in X. Since f is open, $f(f^{-1}(V))$ is open; since f is surjective, $V = f(f^{-1}(V))$. Therefore, V is open.

Hence, f is a quotient map. $\square$

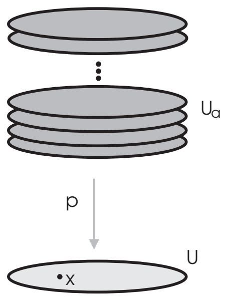
In general, quotient maps do not need to be open.

Covering spaces provide important examples of quotient maps; they are fundamental objects in **algebraic topology**.

Definition. Let $p : E \to B$ be a continuous surjection. p is a **covering map** if every point $x \in B$ has a neighborhood U such that $p^{-1}(U)$ is a disjoint union $\coprod_{a \in A} U_a$ of open sets and $p|_{U_a} : U_a \to U$ is a homeomorphism for all $a \in A$.

A neighborhood U which satisfies this condition is **evenly covered**.

Thus, $p^{-1}(U)$ looks like a stack of pancakes, where each pancake is carried homeomorphically onto U by p.



Lemma. Covering maps are open.

Proof. Let $p : E \to B$ be a covering map, and let $U \subset E$ be open. I want to show that $p(U)$ is open in B.

Take $x \in U$, so $p(x) \in p(U)$. I need to find a neighborhood V of $p(x)$ contained in $p(U)$. Let W be an evenly covered neighborhood of $p(x)$. Since $p^{-1}(W)$ is a disjoint union $\coprod_{a \in A} W_a$ of open sets homeomorphic to W and since $x \in p^{-1}(W)$, x must lie in one of the W_a's.

Thus, let W_a be a neighborhood of x mapped homeomorphically onto W by p. Then $W_a \cap U$ is open, p maps it homeomorphically onto $p(W_a \cap U)$, and $p(x) \in p(W_a \cap U)$. I claim that $p(W_a \cap U)$ is open in B.

To see this, observe that $p|_{W_a} : W_a \to p(W_a) = W$ is a homeomorphism, so the inverse $p|_{W_a}^{-1} : W \to W_a$ is continuous. Now $W_a \cap U$ is open in E, so it's open in W_a. Hence, $(p|_{W_a}^{-1})^{-1}(W_a \cap U) = p(W_a \cap U)$ is open in $p(W_a)$. But $p(W_a) = W$ is open in B, so $p(W_a \cap U)$ is open in B.

I've found a neighborhood $p(W_a \cap U)$ of x which is contained in $p(U)$. Therefore, $p(U)$ is open, and p is an open map. $\square$

Corollary. Covering maps are quotient maps. $\square$

Example. The map $e : \mathbb{R} \to S^1$ given by

$$e(t) = (\cos(2\pi t), \sin(2\pi t))$$

is a covering map; it wraps the real line around the circle. Each open interval of length 1 in $\mathbb{R}$ is mapped homeomorphically to the circle minus a point. $\square$

Example. If $X \times Y$ is a product of topological spaces, the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open, continuous, and surjective. Therefore, they are quotient maps. $\square$

The next result tells you how to construct a map out of a quotient space; it is analogous to the results in algebra which tell you how to construct group maps out of quotient groups, or ring maps out of quotient rings.

Definition. Let $p : X \to Y$ be a quotient map. A **fiber** of p is a set $p^{-1}(y)$ for $y \in Y$.

Theorem. (**Universal Property for Quotients**) Let $p : X \to Y$ be a quotient map. Then any continuous map $f : X \to Z$ which is constant on each fiber of p factors uniquely through p; that is, there is a unique continuous map $\tilde{f} : Y \to Z$ such that $\tilde{f}p = f$.

$$\begin{tikzcd}
X \arrow[d, "p"'] \arrow[dr, "f"] & \\
Y \arrow[r, "\tilde{f}"] & Z
\end{tikzcd}$$

Proof. Let $y \in Y$. Since f is constant on $p^{-1}(y)$, I may take any $x \in p^{-1}(y)$ and define

$$\tilde{f}(y) = f(x).$$

By construction, $\tilde{f}p = f$. I have to show that $\tilde{f}$ is continuous.

Let U be open in Z. I need to show $\tilde{f}^{-1}(U)$ is open in Y. By definition of the quotient topology, $\tilde{f}^{-1}(U)$ is open in Y if and only if $p^{-1}\left(\tilde{f}^{-1}(U)\right)$ is open in X. But since $\tilde{f}p = f$,

$$p^{-1}\left(\tilde{f}^{-1}(U)\right) = f^{-1}(U),$$

which is open in X by continuity of f.

This proves that $\tilde{f}$ is continuous.

To prove uniqueness, suppose that $\tilde{f}_1$ and $\tilde{f}_2$ are continuous maps $Y \to Z$ such that $\tilde{f}_1 p = f$ and $\tilde{f}_2 p = f$. Then $\tilde{f}_1 p = f = \tilde{f}_2 p$; since p is surjective, it is right-cancellable, so $\tilde{f}_1 = \tilde{f}_2$. $\Box$

If you think of the quotient topology as "gluing" or identifying the elements of each fiber, the universal property says that the quotient topology is the "smallest" way of accomplishing this "gluing", in the sense that any other map that accomplishes this "gluing" receives a map from the quotient space.

Corollary. Let $f : X \to Z$ be a continuous surjective map and suppose $p : X \to X^*$ is the quotient map obtained from the partition of X by the fibers $\{f^{-1}(z) \mid z \in Z\}$.

1. There is a continuous bijection $\tilde{f} : X^* \to Z$ such that $\tilde{f}p = f$.
2. $\tilde{f}$ is a homeomorphism if and only if f is a quotient map.

Proof. 1. p is the function which sends $x \in X$ to the partition element $f^{-1}(z) \in X^*$, where $z = f(x)$. f is trivially constant on each fiber, so the Universal Property yields a continuous map $\tilde{f} : X^* \to Z$ such that $\tilde{f}p = f$.

To be explicit, consider a fiber $f^{-1}(z) \in X^*$. Let $x \in f^{-1}(z)$. Then $p(x) = f^{-1}(z)$, so

$$\tilde{f}\left(f^{-1}(z)\right) = \tilde{f}\left(p(x)\right) = f(x) = z.$$

The function $g : Z \to X^*$ given by $g(z) = f^{-1}(z)$ is obviously the inverse of $\tilde{f}$ *as a set map*, so $\tilde{f}$ is bijective. Thus, $\tilde{f}$ is a continuous bijection.

2. If $\tilde{f}$ is a homeomorphism, then $f = \tilde{f}p$ is a composite of quotient maps, so it's a quotient map.

Conversely, suppose f is a quotient map. p is constant on each fiber of f, so the Universal Property applied to the quotient map f yields a continuous map $h : Z \to X^*$ such that $hf = p$.

Now $\tilde{f}hf = f$. That is, $\tilde{f}h$ makes the outer triangle below commute:

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[d, "p"] \arrow[dr, "f"] & \\
Z \arrow[r, "h"] & X^* \arrow[r, "\tilde{f}"] & Z
\end{tikzcd}$$

But the outer triangle clearly commutes if I replace $\tilde{f}h$ with id_Z. By the uniqueness part of the Universal Property applied to the quotient map f, it follows that $\tilde{f}h = \mathrm{id}_Z$.

Likewise, $h\tilde{f}p = p$. In other words, $h\tilde{f}$ makes the outer triangle below commute:

$$\begin{tikzcd}
& X \arrow[dl, "p"'] \arrow[d, "f"] \arrow[dr, "p"] & \\
X^* \arrow[r, "\tilde{f}"] & Z \arrow[r, "h"] & X^*
\end{tikzcd}$$

The outer triangle will commute if I replace $h\tilde{f}$ with id_{X^*}. By the uniqueness part of the Universal Property applied to the quotient map p, it follows that $h\tilde{f} = \mathrm{id}_{X^*}$.

Therefore, $h = \tilde{f}^{-1}$. (In fact, h is the map g defined in the first part of the proof.)

Since $\tilde{f}$ is a continuous bijection with a continuous inverse, it's a homeomorphism. $\square$

Corollary. Let $f : X \to Z$ be a continuous surjective map and suppose $p : X \to X^*$ is the quotient map obtained from the partition of X by the fibers $\{f^{-1}(z) \mid z \in Z\}$. If Z is Hausdorff, then so is X^*.

Proof. Suppose $f^{-1}(z_1)$ and $f^{-1}(z_2)$ are distinct elements of X^*. Then $z_1 \neq z_2$, so I may find disjoint neighborhoods U_1 of z_1 and U_2 of z_2 in Z.

Look at $\tilde{f}^{-1}(U_1)$ and $\tilde{f}^{-1}(U_2)$. These are open in X^*, since $\tilde{f}$ is continuous. Since U_1 and U_2 are disjoint, $\tilde{f}^{-1}(U_1)$ and $\tilde{f}^{-1}(U_2)$ are disjoint.

Finally, $\tilde{f}\left(f^{-1}(z_1)\right) = z_1 \in U_1$ shows that $f^{-1}(z_1) \in \tilde{f}^{-1}(U_1)$, and likewise $f^{-1}(z_2) \in \tilde{f}^{-1}(U_2)$.

Thus, I have disjoint neighborhoods of $f^{-1}(z_1)$ and $f^{-1}(z_2)$ in X^*, so X^* is Hausdorff. $\square$