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Connected Subsets of the Real Line

Proposition. $[0,1]$ is connected in the standard topology.

Proof. Suppose $[0,1] = U \cup V$ is a separation, so U and V are nonempty, disjoint, open sets.

The idea of the proof is to find a point on the "boundary" between U and V, then show that such a point can't be in either U or V.

Without loss of generality, suppose that $0 \in U$. Let

$$S = \{s \in [0,1] \mid [0,s] \subset U\}.$$

Since $0 \in S$, S is nonempty. Moreover, since $0 \in U$ and U is open, there is an $\epsilon > 0$ such that $[0,\epsilon) \subset U$. Then $\left[0, \dfrac{\epsilon}{2}\right] \subset U$, so $\dfrac{\epsilon}{2} \in S$. Thus, $S \neq \{0\}$.

S is a nonempty set which is bounded above, so it has a least upper bound. Let

$$s_0 = \sup S.$$

Since $S \neq \{0\}$, $s_0 > 0$.

s_0 is the sort of "boundary point" I alluded to above. Before using it to obtain a contradiction, I'll prove two preliminary results.

Claim 1: If $s \in S$ and $0 \le r < s$, then $r \in S$.

$$[0,r] \subset [0,s] \subset U, \quad \text{since} \quad s \in S.$$

Therefore, $r \in S$.

Claim 2: If $0 \le s < s_0$, then $s \in S$ and $s \in U$.

Suppose $0 \le s < s_0$ but $s \notin S$. If $t > s$, then $t \notin S$, else $s \in S$ by Claim 1. Since no element of S is greater than s, it follows that s is an upper bound for S. But $s < s_0$, and s_0 is the *least* upper bound for S. This contradiction implies that $s \in S$; thus, $[0,s] \subset U$, so $s \in U$.

Finally, I'll obtain a contradiction by showing that s_0 can't be in S or its complement.

Suppose $s_0 \in S$, so $[0,s_0] \subset U$. Note that $s_0 < 1$ — for if $s_0 = 1$, then $[0,1] = [0,s_0] \subset U$, contradicting the fact that V is nonempty.

Now $0 < s_0 < 1$, $s_0 \in U$, and U is open, so there is an $\epsilon > 0$ such that $(s_0 - \epsilon, s_0 + \epsilon) \subset U$. Hence,

$$\left[0, s_0 + \frac{\epsilon}{2}\right] = [0,s_0] \cup \left[s_0, s_0 + \frac{\epsilon}{2}\right] \subset U,$$

so $s_0 + \dfrac{\epsilon}{2} \in S$. This contradicts the fact that s_0 is the least upper bound for S.

It follows that $s_0 \notin S$. Therefore, $[0,s_0] \not\subset U$. By Claim 2, $[0,s_0) \subset U$, so $s_0 \notin U$. Therefore, $s_0 \in V$. Now $s_0 > 0$ and V is open, so there is an $\epsilon > 0$ such that $(s_0 - \epsilon, s_0] \subset V$. In particular, $s_0 - \dfrac{\epsilon}{2} \in V$, and this contradicts Claim 2.

It follows that there is no such separation, and hence $[0,1]$ is connected. $\square$

Corollary. Any closed interval in $\mathbb{R}$ is connected.

Proof. Any closed interval in $\mathbb{R}$ is homeomorphic to $[0,1]$. $\square$

Corollary. $\mathbb{R}$ is connected.

Proof.

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$$

represents $\mathbb{R}$ as a union of connected sets having points in common (specifically, the points in $[-1, 1]$ are common to all of the intervals). By an earlier result, this proves that $\mathbb{R}$ is connected. □

Corollary. $\mathbb{R}^n$ is connected.

Proof. $\mathbb{R}^n$ is a product of connected spaces, so it's connected. □

Corollary. Any open interval in $\mathbb{R}$ is connected.

Proof. Any open interval in $\mathbb{R}$ is homeomorphic to $\mathbb{R}$. □

In fact, more is true. Let L be an ordered set with at least two elements. L is a **linear continuum** if L has the least upper bound property (every subset bounded above has a least upper bound) and if $x < y$ implies there exists z such that $x < z < y$. For example, $\mathbb{R}$ is a linear continuum. One may show that every linear continuum is connected.

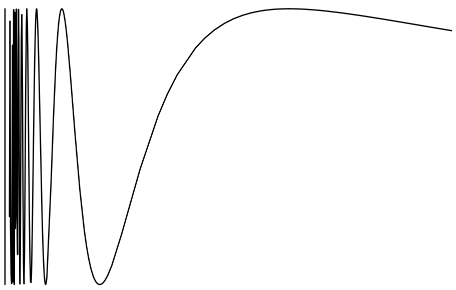
You can show that many sets in $\mathbb{R}^n$ are connected by using the results above and general stuff about connected sets.

Example. S^1 is a connected subset of $\mathbb{R}^2$.

To see this, note that the continuous function $f : \mathbb{R} \to S^1$ given by $f(t) = (\cos t, \sin t)$ maps the connected set $\mathbb{R}$ onto S^1.

More generally, if $f : \mathbb{R} \to \mathbb{R}^2$ is continuous, then the image is connected. For example, the graph of a function $y = f(x)$ defined for all $x \in \mathbb{R}$ is a connected set. □

Example. The **topologist's sine curve** consists of the graph of $y = \sin \dfrac{1}{x}$ for $0 < x \leq 1$. The **closed topologist's sine curve** consists of the union of the topologist's sine curve with the segment $\{(0, y) \mid -1 \leq y \leq 1\}$.



The topologist's sine curve is connected, because it's a continuous image of a connected set. The closed topologist's sine curve is the closure of the topologist's sine curve in $\mathbb{R}^2$, so it's connected as well. □

Here's a familiar result from calculus.

Theorem. (**Intermediate Value Theorem**) Let $f : X \to Y$ be a continuous function, where X is connected and Y is an ordered set with the order topology. Let $a, b \in X$, and let d be an element of Y between $f(a)$ and $f(b)$ (so $f(a) \leq d \leq f(b)$ or $f(b) \leq d \leq f(a)$). Then $d = f(c)$ for some $c \in X$.

Proof. If $d = f(a)$ or $d = f(b)$, then I'm done. So assume $d \neq f(a), f(b)$. Then $f(a) < d < f(b)$ or $f(b) < d < f(a)$.

$f(X)$ is connected, since X is connected. Consider the sets $f(X) \cap (-\infty, d)$ and $f(X) \cap (d, +\infty)$. These are disjoint open subsets of $f(X)$; they are nonempty, since one contains $f(a)$ and the other contains $f(b)$. Therefore, their union isn't all of $f(X)$. Hence, $f(X) \cap \{d\}$ is nonempty.

An element $f(X) \cap \{d\}$ is an element $f(c)$ for $c \in X$ such that $f(c) = d$. $\square$