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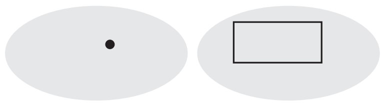
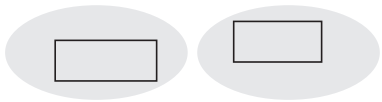
The Separation Axioms

Definition. Let X be a space in which singletons are closed.

1. X is **regular** if a closed set and a point outside it can be separated by disjoint open sets.
2. X is **normal** if disjoint closed sets can be separated by disjoint open sets.



regular



normal

Obviously, a normal space is regular, and a regular space is Hausdorff.

Terminology. Recall that a Hausdorff space is also known as a T_2 space.

X is a T_0 space if given distinct points x, y, there is a neighborhood U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

X is a T_1 space if given distinct points x, y, there are neighborhoods U of x and V of y such that $y \notin U$ and $x \notin V$.

X is a $T_{2\frac{1}{2}}$ space if distinct points have neighborhoods whose closures are disjoint.

If you don't require that points be closed, then a space in which disjoint points and closed sets can be separated is T_3, and a space in which disjoint closed sets can be separated is T_4.

Lemma. Let X be a space in which singletons are closed.

1. X is regular if and only if for all $x \in X$ and every neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$.
2. X is normal if and only if for every closed subset A of X and every open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof. 1. Suppose X is regular, $x \in X$, and U is a neighborhood of x. Then $X - U$ is a closed set disjoint from x, so by regularity I may find disjoint open sets V containing x and W containing $X - U$.

In particular, $x \in V \subset X - U$; since $X - U$ is closed, $\overline{V} \subset X - U$.

Conversely, suppose that for all $x \in X$ and every neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$. Let $x \in X$ and let C be a closed set which does not contain x.

$X - C$ is an open set containing x, so there is a neighborhood V of x such that $\overline{V} \subset X - C$. Then $X - \overline{V}$ is an open set containing C, and it is disjoint from the open set V containing x. Therefore, X is regular.

2. The proof for normality is the same as the proof for regularity, with the point x replaced with the closed set A. $\square$

Corollary. If X is a regular space, then X is a $T_{2\frac{1}{2}}$ space.

Proof. Suppose X is regular. Let x and y be distinct points in X. Points in X are closed, so $X - \{y\}$ is a neighborhood of x. By the preceding result, there is a neighborhood U of x such that $\overline{U} \subset X - \{y\}$. Again by the preceding result, there is a neighborhood V of x such that $\overline{V} \subset U$.

$X - \overline{U}$ is a neighborhood of y, and $X - \overline{U} \subset X - U$. Since $X - U$ is closed, $\overline{X - \overline{U}} \subset X - U$. Since $\overline{V} \subset U$ and $\overline{X - \overline{U}} \subset X - U$, $\overline{V}$ and $\overline{X - \overline{U}}$ are disjoint.

Thus, V is a neighborhood of x, $X - \overline{U}$ is a neighborhood of y, and their closures are disjoint. Therefore, X is $T_{2\frac{1}{2}}$. $\square$

Corollary. If X is a locally compact Hausdorff space, then X is regular. $\square$

Example. (Irrational slope topology [1, Example 75]) Let

$$X = \mathbb{Q} \times \mathbb{Q}^{\geq 0},$$

the points with rational coordinates in the closed upper half-plane. Fix an irrational number θ.

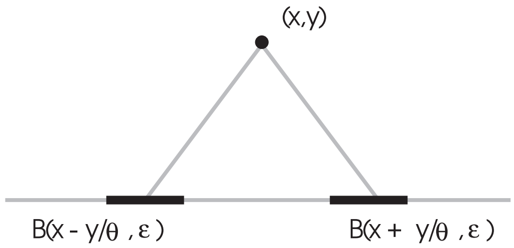
If $x, \epsilon \in \mathbb{R}$ and $\epsilon > 0$, define a subset of the x-axis $B(x; \epsilon)$ by

$$B(x; \epsilon) = \{y \in \mathbb{Q} \mid |x - y| < \epsilon\}.$$

Then for $(x, y) \in X$ and $\epsilon > 0$, set

$$N((x, y), \epsilon) = \{(x, y)\} \cup B\left(x - \frac{y}{\theta}, \epsilon\right) \cup B\left(x + \frac{y}{\theta}, \epsilon\right).$$

$N((x, y), \epsilon)$ consists of the point (x, y) together with two open ϵ-intervals of rationals on the x-axis. The intervals are centered at points on the x-axis which determine lines of slopes θ and $-\theta$ with the point (x, y).



Note that if $y = 0$, $N((x, y), \epsilon)$ reduces to an open interval of rationals about x in the x-axis. Moreover, if $x \in \mathbb{Q}$, then $N((x, 0), \epsilon)$ is a basic open set.

Define a topology by taking as a basis all the sets $N((x, y), \epsilon)$ for all real $\epsilon > 0$ and all $(x, y) \in X$. It's clear that the sets cover X.

If $(a, b) \in N((x, y), \epsilon_1) \cap N((x, y), \epsilon_2)$, then either $(a, b) = (x, y)$ or (a, b) lies in the intersection of the intervals in the x-axis.

In the first case, if $\epsilon = \min(\epsilon_1, \epsilon_2)$, then $N((x, y), \epsilon)$ is a basis element containing (a, b) and contained in $N((x, y), \epsilon_1) \cap N((x, y), \epsilon_2)$.

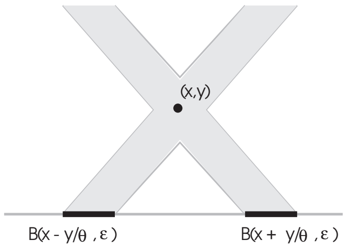
In the second case, $(a, b) = (a, 0)$, and $(a, 0) \in B\left(x \pm \frac{y}{\theta}, \epsilon_1\right) \cap B\left(x \pm \frac{y}{\theta}, \epsilon_2\right)$, and this is the intersection of open intervals in the x-axis. Choosing a sufficiently small $\epsilon > 0$, I may find an interval $N((a, 0), \epsilon)$ contained in this intersection.

If $(a, b) \in N((x_1, y_1), \epsilon_1) \cap N((x_2, y_2), \epsilon_2)$ where $(x_1, y_1) \neq (x_2, y_2)$, then again (a, b) must lie in the intersection of intervals in the x-axis. As before, I may find a basic interval containing x contained in the intersection of the original intervals.

Therefore, the collections of sets $N((x, y), \epsilon)$ for all real $\epsilon > 0$ and all $(x, y) \in X$ forms a basis.

Next, I'll show that the closures of any two basic neighborhoods must intersect.

What is $\overline{N((x, y), \epsilon)}$? It consists of two diagonal strips of slopes θ and $-\theta$ emanating from the intervals $B\left(x - \frac{y}{\theta}, \epsilon\right)$ and $B\left(x + \frac{y}{\theta}, \epsilon\right)$.



To see this, observe that a basic neighborhood of a point in these strips will contain intervals in the x-axis which intersect $B\left(x-\dfrac{y}{\theta},\epsilon\right)$ or $B\left(x+\dfrac{y}{\theta},\epsilon\right)$.

It is clear that any two such pairs of strips must intersect. Therefore, the closures of any two basic neighborhoods must intersect. By the result above, X is not regular. $\square$

I proved a special case of the first part of the following proposition; I'm repeating the proof of the second part for the sake of completeness.

Proposition.

1. A product of Hausdorff spaces is Hausdorff.
2. A subspace of a Hausdorff space is Hausdorff.

Proof. 1. Let $\{X_a\}_{a\in A}$ be a family of Hausdorff spaces. Let $(x_a), (y_a)$ be distinct points in $\prod_{a\in A} X_a$. For some index b, $x_b \neq y_b$. Choose disjoint open sets $U, V \subset X_b$ such that $x_b \in U$ and $y_b \in V$.

Define

$$U_a = \begin{cases} X_a & \text{if } a \neq b \\ U & \text{if } a = b \end{cases} \quad \text{and} \quad V_a = \begin{cases} X_a & \text{if } a \neq b \\ V & \text{if } a = b \end{cases}.$$

$\prod_{a\in A} U_a$ is a neighborhood of (x_a), $\prod_{a\in A} V_a$ is a neighborhood of (y_a), and $\prod_{a\in A} U_a$ and $\prod_{a\in A} V_a$ are disjoint. Therefore, $\{X_a\}_{a\in A}$ is Hausdorff.

2. Let X be a Hausdorff space, and let $Y \subset X$. Let $x, y \in Y$. Find disjoint neighborhoods U of x and V of y in X. Then $U \cap Y$ is a Y-neighborhood of x, $V \cap Y$ is a Y-neighborhood of y, and $U \cap Y$ and $V \cap y$ are disjoint. Therefore, Y is Hausdorff. $\square$

Proposition.

1. A product of regular spaces is regular.
2. A subspace of a regular space is regular.

Proof. 1. Let $\{X_a\}_{a\in A}$ be a family of regular topological spaces. For each a, X_a is Hausdorff. Therefore, $\prod_{a\in A} X_a$ is Hausdorff, so points in $\prod_{a\in A} X_a$ are closed.

Let (x_a) be a point in $\prod_{a\in A} X_a$. Take a neighborhood of (x_a) in $\prod_{a\in A} X_a$; replacing the neighborhood with a smaller one if necessary, I may assume that it has the form $\prod_{a\in A} U_a$, where U_a is open in X_a and $U_a = X_a$ for all but finitely many a's.

Since X_a is regular for each a, I may find a neighborhood V_a of x_a in X_a such that $\overline{V_a} \subset U_a$. If a is an index for which $U_a = X_a$, I will take $V_a = X_a$ as well. Then $\prod_{a\in A} V_a$ is a neighborhood of (x_a) in $\prod_{a\in A} X_a$, and

$$\overline{\prod_{a\in A} V_a} = \prod_{a\in A} \overline{V_a} \subset \prod_{a\in A} U_a.$$

It follows by an earlier result that $\prod_{a\in A} X_a$ is regular.

2. Let X be a regular topological space, and let $Y \subset X$. A point of Y is closed in X, so it is closed in Y.

Let $x \in Y$, and let C be a closed subset of Y which does not contain x. Write $C = Y \cap D$, where D is closed in X. Find disjoint open sets U and V in X such that $x \in U$ and $D \subset V$. Then $Y \cap U$ and $Y \cap V$ are disjoint open sets in Y, $x \in Y \cap U$, and $C \subset Y \cap V$. Therefore, Y is regular. $\square$

Remark. The preceding results are false for normal spaces: A subspace of a normal space need not be normal, and a product of normal spaces need not be normal.

The following results say that the class of normal spaces is large enough to contain many interesting spaces: specifically, metric spaces, compact Hausdorff spaces, and regular spaces having countable bases.

Theorem. Every metric space is normal.

Proof. Let (X, d) be a metric space. Since metric spaces are Hausdorff, points in X are closed.

Let C and D be disjoint closed sets in X. Since $C \subset X - D$ and $X - D$ is open, for every $x \in C$ I may find an open ball $B(x; \epsilon_x)$ contained in $X - D$.

By a similar argument, for every $y \in D$, there is a ball $B(y; \epsilon_y)$ contained in $X - C$.

Let

$$U = \bigcup_{x \in C} B\left(x; \frac{\epsilon_x}{2}\right) \quad \text{and} \quad V = \bigcup_{y \in D} B\left(y; \frac{\epsilon_y}{2}\right).$$

U and V are open sets, $C \subset U$, and $D \subset V$. I claim that U and V are disjoint.

Suppose on the contrary that $z \in U \cap V$ — say $z \in B\left(x; \frac{\epsilon_x}{2}\right)$ for $x \in C$ and $z \in B\left(y; \frac{\epsilon_y}{2}\right)$ for $y \in D$. Then

$$d(x, y) \le d(x, z) + d(y, z) < \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2}.$$

If $\epsilon_x \le \epsilon_y$, then

$$d(x, y) < \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2} \le \frac{\epsilon_y}{2} + \frac{\epsilon_y}{2} = \epsilon_y.$$

This means that $x \in B(y; \epsilon_y)$, so $x \notin C$, contradicting the fact that $x \in C$.

Likewise, $\epsilon_y \le \epsilon_x$ implies that $d(x, y) < \epsilon_x$, which in turns implies that $y \in B(x; \epsilon_x)$. This means that $y \notin D$, contradicting the fact that $y \in D$.

This proves my claim that U and V are disjoint. Since I've separated the closed sets C and D with disjoint open sets U and V, it follows that X is normal. $\square$

Theorem. A compact Hausdorff space is normal.

Proof. In a Hausdorff space, points are closed.

I need to show that disjoint closed sets can be separated by disjoint open sets. First, recall that in a Hausdorff space, a point and a compact set that doesn't contain it can be separated by disjoint open sets.

Let C and D be disjoint closed subsets of the compact Hausdorff space X. Since D is closed and X is compact, D is compact. By the observation above, for each $x \in C$ I may find disjoint open sets U_x and V_x such that $x \in U_x$ and $D \subset V_x$.

$\{U_x\}_{x \in C}$ is an open cover of C; C is compact, since it's a closed subset of a compact space. Let $\{U_{x_1}, \ldots, U_{x_n}\}$ be a finite subcover of C. Define

$$U = \bigcup_{i=1}^{n} U_{x_i} \quad \text{and} \quad V = \bigcap_{i=1}^{n} V_{x_i}.$$

U and V are open sets, $C \subset U$, and $D \subset V$. If $y \in U \cap V$, then $y \in U_{x_i}$ for some i and $y \in V \subset V_{x_i}$, so $y \in U_{x_i} \cap V_{x_i} = \emptyset$. This contradiction shows that U and V are disjoint.

Therefore, X is normal. $\square$

Theorem. A regular space with a countable basis is normal.

Proof. Let X be a regular space with a coutable basis $\{B_i\}_{i \in \mathbb{Z}^+}$. Since X is regular, points are closed.

Let C and D be disjoint closed subsets of X. Let $x \in C$. Since $X - D$ is an open set containing x, I may find a neighborhood U of x such that $x \in U \subset X - D$. Next, regularity implies that there is a neighborhood V of x such that $\overline{V} \subset U$. Finally, there is a basis element B_i containing x such that $B_i \subset V$. Now $\overline{B_i} \subset \overline{V} \subset U \subset X - D$, so $\overline{B_i}$ misses D.

Repeat this procedure for each $x \in C$. I wind up up with a countable subcollection $\{C_j\}$ of $\{B_i\}$ which covers C, and which satisfies $\overline{C_j} \cap D = \emptyset$ for all j.

Likewise, I may find a countable subcollection $\{D_k\}$ of $\{B_i\}$ which covers D, and which satisfies $\overline{D_k} \cap C = \emptyset$ for all k.

What I'd like to do is to take the unions of the two subcollections as my neighborhoods of C and D, but these unions may not be disjoint. However, since the subcollections are countable, I can inductively adjust the subcollections to produce new subcollections whose unions will be disjoint.

Thus, for each j and k, define

$$C'_j = C_j - \bigcup_{k=1}^{j} \overline{D_k} \quad \text{and} \quad D'_k = D_k - \bigcup_{j=1}^{k} \overline{C_j}.$$

Then let

$$A = \bigcup_{j=1}^{\infty} C'_j \quad \text{and} \quad B = \bigcup_{k=1}^{\infty} D'_k.$$

I claim that A and B are disjoint open sets, $C \subset A$, and $D \subset B$.

Each C'_j and each D'_k is open, since each is an open set minus a closed set. Therefore, A and B are unions of open sets, so they are open.

The union of the C_j's contains C. This is not changed by subtracting $\overline{D_k}$'s, because these sets missed C anyway. Therefore, $A = \bigcup_{j=1}^{\infty} C'_j$ contains C. Similarly, B contains D.

Finally, I'll show that A and B are disjoint. Suppose $y \in A \cap B$. Then $y \in C'_j \cap D'_k$ for some j and k. Without loss of generality, suppose that $j \le k$. Now $y \in C'_j$ implies $y \in C_j$, but $y \in D'_k$ implies $y \notin \overline{C_j}$ for $j \le k$. This contradiction proves that $A \cap B = \emptyset$.

Therefore, A and B are disjoint neighborhoods of C and D, and X is normal. $\square$

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[1] Lynn A. Steen and J. Arthur Seebach, *Counterexamples in Topology.* New York: Holt, Rinehart, and Winston, Inc., 1970. [ISBN 0-03-079485-4]