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The Tietze Extension Theorem

The **Tietze Extension Theorem** says that a continuous real-valued function on a closed subset of a normal space may be extended to the entire space. I'll show that this extension condition is essentially *equivalent* to normality.

The construction of the extension involves building a sequence of functions whose *sum* agrees with the given function on the subspace.

Lemma. Let X be a normal space, let A be a closed subset of X, and let $f : A \to \mathbb{R}$ be a continuous function satisfying $|f(x)| \le c$ for all $x \in A$ and some $c \in \mathbb{R}$. Then there is a continuous function $g : X \to \mathbb{R}$ such that

1. $|g(x)| \le \dfrac{1}{3}c$ for all $x \in X$.
2. $|f(x) - g(x)| \le \dfrac{2}{3}c$ for all $x \in A$.

Proof. The sets

$$f^{-1}\left(\left[\frac{1}{3}c, \infty\right)\right) \quad \text{and} \quad f^{-1}\left(\left(-\infty, -\frac{1}{3}c\right]\right)$$

are disjoint closed sets in A; since A is closed, these sets are also closed in X. By Urysohn's lemma, I may find a continuous function $g : X \to \left[-\dfrac{1}{3}c, \dfrac{1}{3}c\right]$ such that

$$g\left(f^{-1}\left(\left(-\infty, -\frac{1}{3}c\right]\right)\right) = -\frac{1}{3}c \quad \text{and} \quad g\left(f^{-1}\left(\left[\frac{1}{3}c, \infty\right)\right)\right) = \frac{1}{3}c.$$

By construction, $|g(x)| \le \dfrac{1}{3}c$ for all $x \in X$.

Now consider cases. If $-c \le f(x) \le -\dfrac{1}{3}c$, then $g(x) = -\dfrac{1}{3}c$, so $|f(x) - g(x)| \le \dfrac{2}{3}c$.

If $\dfrac{1}{3}c \le f(x) \le c$, then $g(x) = \dfrac{1}{3}c$, so $|f(x) - g(x)| \le \dfrac{2}{3}c$.

Finally, suppose $-\dfrac{1}{3}c < f(x) < \dfrac{1}{3}c$. Since $-\dfrac{1}{3}c \le g(x) \le \dfrac{1}{3}c$, I again have $|f(x) - g(x)| \le \dfrac{2}{3}c$. $\square$

Theorem. Let X be Hausdorff. The following are equivalent:

1. X is normal.
2. If A is a closed subset of X, any continuous function $f : A \to \mathbb{R}$ extends to a continuous function $F : X \to \mathbb{R}$.

   To say that F **extends** f means that $F|_A = f$.

Remark. $\mathbb{R}$ may be replaced by a closed interval $[a, b]$.

Proof. ($2 \Rightarrow 1$) Suppose that if A is a closed subset of X, any continuous function $f : A \to \mathbb{R}$ extends to a continuous function $F : X \to \mathbb{R}$. Let C and D be disjoint closed subsets of X. Define $f : C \cup D \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in C \\ 1 & \text{if } x \in D \end{cases}.$$

$C \cup D$ is closed, so by assumption, I may extend f to a function $F : X \to \mathbb{R}$. Pick disjoint open sets U and V in $\mathbb{R}$ such that $0 \in U$ and $1 \in V$. Then $F^{-1}(U)$ and $F^{-1}(V)$ are disjoint open sets in X, $C \subset F^{-1}(U)$, and $D \subset F^{-1}(V)$. Therefore, X is normal.

$(1 \Rightarrow 2)$ Suppose first that $|f(x)| \leq c$ for all $x \in A$ and some $c \in \mathbb{R}$. I'll extend f to $F : X \to \mathbb{R}$ such that $|F(x)| \leq c$.

By the Lemma, I may find a continuous function $g_0 : X \to \mathbb{R}$ such that

$$\begin{aligned} |g_0(x)| &\leq \frac{1}{3}c \quad \text{for all} \quad x \in X, \\ |f(x) - g_0(x)| &\leq \frac{2}{3}c \quad \text{for all} \quad x \in A. \end{aligned}$$

Next, apply the Lemma to $f - g_0 : A \to \mathbb{R}$ to find a function $g_1 : X \to \mathbb{R}$ such that

$$\begin{aligned} |g_1(x)| &\leq \frac{1}{3} \cdot \frac{2}{3}c \quad \text{for all} \quad x \in X, \\ |f(x) - g_0(x) - g_1(x)| &\leq \frac{2}{3} \cdot \frac{2}{3}c \quad \text{for all} \quad x \in A. \end{aligned}$$

Suppose $n > 1$ and $g_{n-1} : X \to \mathbb{R}$ has been defined and satisfies

$$\begin{aligned} |g_{n-1}(x)| &\leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} c \quad \text{for all} \quad x \in X, \\ |f(x) - g_0(x) - g_1(x) - \cdots - g_{n-1}(x)| &\leq \left(\frac{2}{3}\right)^{n} c \quad \text{for all} \quad x \in A. \end{aligned}$$

Apply the Lemma to find $g_n : X \to \mathbb{R}$ satisfying

$$\begin{aligned} |g_n(x)| &\leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n} c \quad \text{for all} \quad x \in X, \\ |f(x) - g_0(x) - g_1(x) - \cdots - g_n(x)| &\leq \left(\frac{2}{3}\right)^{n+1} c \quad \text{for all} \quad x \in A. \end{aligned}$$

I want to define $F(x) = \displaystyle\sum_{n=0}^{\infty} g_n(x)$. I must show that this series converges. To see this, note that $|g_n(x)| \leq \dfrac{1}{3} \cdot \left(\dfrac{2}{3}\right)^n c$, and that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n c$ is a convergent geometric series. Therefore, $\sum_{n=0}^{\infty} g_n(x)$ converges absolutely by direct comparison.

Next, I want to show that $F(x)$ is continuous. Let $x \in X$, and let

$$s_k(x) = \sum_{n=0}^{k} g_n(x)$$

be the k^{th} partial sum.

If $k > j$, then

$$|s_k(x) - s_j(x)| = \left|\sum_{n=0}^{k} g_n(x) - \sum_{n=0}^{j} g_n(x)\right| = \left|\sum_{n=j+1}^{k} g_n(x)\right| \leq \sum_{n=j+1}^{k} |g_n(x)| \leq \sum_{n=j+1}^{k} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n c =$$

$$\frac{1}{3} \cdot \left(\frac{2}{3}\right)^{j+1} c \cdot \sum_{n=0}^{k-(j+1)} \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{j+1} c \cdot \left(\frac{1 - \left(\frac{2}{3}\right)^{k-j}}{1 - \frac{2}{3}}\right) = \left(\left(\frac{2}{3}\right)^{j+1} - \left(\frac{2}{3}\right)^{k+1}\right) c.$$

Fix j and let $k \to \infty$:

$$|F(x) - s_j(x)| \le \left(\frac{2}{3}\right)^{j+1} c.$$

Since the right side goes to 0 as $j \to \infty$ *independent of* x, it follows that the partial sums converge uniformly to $F(x)$. Since the partial sums are continuous, it follows that $F(x)$ is continuous.

For $x \in A$, I have

$$\left|f(x) - \sum_{n=0}^{k} g_n(x)\right| \le \left(\frac{2}{3}\right)^{k+1} c.$$

As $k \to \infty$, the right side goes to 0, and the sum goes to $F(x)$. Thus, F agrees with f on A.

Before I consider the case where f is unbounded, I need to observe something about the preceding construction. I claim that if $|f(x)| < c$, I can construct an extension $F'(x)$ satisfying $|F'(x)| < c$.

To see this, first construct $F(x)$ as above. Note that

$$|F(x)| = \left|\sum_{n=0}^{\infty} g_n(x)\right| \le \sum_{n=0}^{\infty} |g_n(x)| \le \sum_{n=0}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n c = c.$$

Let $B = F^{-1}(\{-c\} \cup \{c\})$. This is a closed set disjoint from A, since if $x \in A$, $|F(x)| = |f(x)| < c$. Let ϕ be a continuous function such that $\phi(A) = 1$ and $\phi(B) = 0$. Define $F'(x) = \phi(x) \cdot F(x)$. F' still extends f, but now $|F'(x)| < c$.

Now I'll consider the unbounded case where $f : A \to \mathbb{R}$.

Let $\psi : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be the arctangent function $\psi(x) = \arctan x$. Then $\psi \cdot f : A \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfies $|\psi \cdot f(x)| < \frac{\pi}{2}$. Construct an extension $F : X \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $\psi^{-1} \cdot F : X \to \mathbb{R}$ is a continuous function which extends f. $\square$