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Topological Spaces

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = a$ if and only if

For every $\epsilon > 0$, there is a $\delta > 0$ such that if $\delta > |x - a| > 0$, then $\epsilon > |f(x) - f(a)|$.

In words,

You can make $f(x)$ arbitrarily close to $f(a)$ by making x sufficiently close to a.

Continuity is important in analysis and calculus. The idea of "closeness" is an important part of the definition of continuity. What would "closeness" mean in an abstract setting — one in which there's no way to measure the *distance* between two points?

A **topology** for a space is, roughly speaking, a specification of all "closeness relations" in the space.

Definition. Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X satisfying the following conditions:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. The union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.
3. The *finite* intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.

The elements of $\mathcal{T}$ are called **open sets**. The set X together with the topology $\mathcal{T}$ is a **topological space**.

Example. If X is a set, take $\mathcal{D}$ to be $\mathcal{P}(X)$, the power set of X. (Recall that $\mathcal{P}(X)$ is the set of all subsets of X.)

$\mathcal{D}$ is clearly a topology on X; it is called the **discrete topology**. In the discrete topology, all subsets are open. $\square$

The other extreme is the **indiscrete topology** on X. In this case, $\mathcal{I} = \{\emptyset, X\}$. Again, it is clear that this is a topology on X. $\square$

Definition. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X, then $\mathcal{T}_1$ is **finer** than $\mathcal{T}_2$ (or $\mathcal{T}_2$ is **coarser** than $\mathcal{T}_1$) if $\mathcal{T}_1 \supset \mathcal{T}_2$. In other words, finer topologies contain more open sets.

Thus, if $\mathcal{D}$ is the discrete topology on X, then $\mathcal{D}$ is finer than any other topology on X. If $\mathcal{I}$ is the indiscrete topology on X, then $\mathcal{I}$ is coarser than any other topology on X.

It is, of course, possible for two topologies on X to be incomparable, i.e. neither may be finer than the other.

Analysts often refer to "stronger" and "weaker" topologies, but their usage is often at variance with the usage of topologists.

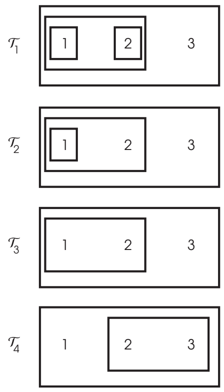
Example. Let $X = \{1, 2, 3\}$, and consider the following topologies on X:

$$\mathcal{T}_1 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\},$$

$$\mathcal{T}_2 = \{\emptyset, \{1\}, \{1, 2\}, X\},$$

$$\mathcal{T}_3 = \{\emptyset, \{1, 2\}, X\}.$$

$$\mathcal{T}_4 = \{\emptyset, \{2, 3\}, X\}.$$



Then $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, and $\mathcal{T}_2$ is finer than $\mathcal{T}_3$. On the other hand, $\mathcal{T}_4$ is not comparable with the other three.

As this example shows, topology does not necessarily concern itself with the concepts of measurement or distance. They might not be applicable to the space under consideration. □

Example. The standard topology on $\mathbb{R}$ is the topology whose open sets consist of arbitrary unions of open intervals.

The empty set is considered to be a degenerate open interval, and $\mathbb{R}$ can be written as a union of open intervals.

A union of unions of open intervals is a union of open intervals.

It remains to show that this topology is closed under finite intersections. It's fairly obvious that two open intervals either do not intersect, or intersect in an open interval; I won't write out the details. The general case for finite intersections reduces to this case by distributing intersections over unions. □

Example. The standard topology on $\mathbb{R}^n$ is obtained by starting with open balls

$$B(x; r) = \{y \in \mathbb{R}^n \mid |x - y| < r\}.$$

($B(x; r)$ is called the **open r-ball centered at x**.) The open sets are arbitrary unions of open balls.

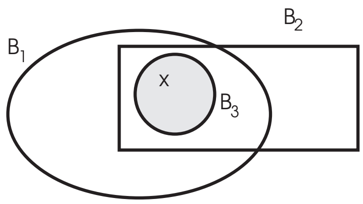
I'll defer the verification that this gives a topology, since it will follow easily once I develop the notion of a **basis**. □

The examples above give a typical way of defining a topology: Start with a collection of subsets and then extend the collection to a topology by taking arbitrary unions. The next definition makes precise the conditions I need to be able to do this.

Definition. Let X be a set. A collection $\mathcal{B}$ of subsets of X is a **basis** if:

1. Every element of X is contained in some element of $\mathcal{B}$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is an element $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$



The elements of $\mathcal{B}$ are called **basic subsets**, or **basis elements**.

Remark. By induction, it follows that if $\mathcal{B}$ is a basis, $B_1, \ldots, B_n \in \mathcal{B}$, and $x \in B_1 \cap \cdots \cap B_n$, then there is an element $B \in \mathcal{B}$ such that

$$x \in B \subset B_1 \cap \cdots \cap B_n.$$

Example. The open intervals in $\mathbb{R}$ form a basis.

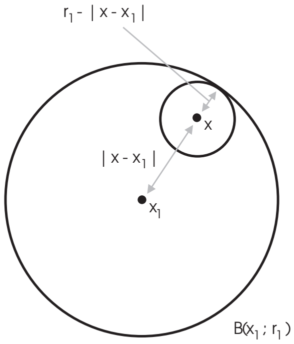
If $x \in \mathbb{R}$, then $x \in (x-1, x+1)$ (for instance).

If x is contained in the intersection of two open intervals, the intersection is itself an open interval which contains x and is contained in the intersection.

The open balls $B(x; r)$ in $\mathbb{R}^n$ form a basis. Obviously, every point in $\mathbb{R}^n$ is contained in some open ball.

Take open balls $B(x_1; r_1)$ and $B(x_2; r_2)$. Assume that they intersect, and let $x \in B(x_1; r_1) \cap B(x_2; r_2)$. I have to find an open ball centered at x which is contained in the intersection.

Let $s_1 = r_1 - |x - x_1|$. Then $B(x; s_1) \subset B(x_1, r_1)$.



Likewise, let $s_2 = r_2 - |x - x_2|$. Then $B(x; s_2) \subset B(x_2, r_2)$.

Therefore, if I let $r = \min(s_1, s_2)$, then

$$B(x; r) \subset B(x_1; r_1) \cap B(x_2; r_2).$$

This shows that the open balls in $\mathbb{R}^n$ form a basis. $\square$

Proposition. Let X be a set, and let $\mathcal{B}$ be a basis. Define $\mathcal{T}$ to be the collection of subsets $U \subset X$ satisfying the following property:

$U \in \mathcal{T}$ if and only if for all $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

Then $\mathcal{T}$ is a topology on X.

$\mathcal{T}$ is the **topology generated by the basis $\mathcal{B}$**.

Proof. The empty set is in $\mathcal{T}$; it satisfies the defining condition vacuously, since it has no elements.

$X \in \mathcal{T}$: If $x \in X$, the first axiom for a basis implies that there is an element $B \in \mathcal{B}$ such that $x \in B \subset X$.

Let $\{U_a\}$ be a collection of elements of $\mathcal{T}$. I need to show that the union $U = \bigcup_a U_a$ is in $\mathcal{T}$.

Let $x \in U$. Then $x \in U_a$ for some a. Since $U_a \in \mathcal{T}$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U_a$. Then

$$x \in B \subset U_a \subset U.$$

Hence, $U \in \mathcal{T}$.

Let $V_1, \ldots, V_n \in \mathcal{T}$. I must show that $V = \bigcap_{i=1}^n V_i$ is in $\mathcal{T}$. Let $x \in V$. Then $x \in V_i$ for all i. Since $V_i \in \mathcal{T}$, for each i I may find an element $B_i \in \mathcal{B}$ such that $x \in B_i \subset V_i$. By the remark above, I may find an element $B \in \mathcal{B}$ such that $x \in B \subset \bigcap_{i=1}^n B_i$. Then

$$x \in B \subset \bigcap_{i=1}^n B_i \subset \bigcap_{i=1}^n V_i = V.$$

Hence, $V \in \mathcal{T}$.

Therefore, $\mathcal{T}$ is a topology. $\square$

Notice that elements of a basis are automatically elements of the topology they generate: Basic sets are open.

Definition. If $\mathcal{T}$ is a topology and $\mathcal{B}$ is a basis, then $\mathcal{B}$ is a **basis for** $\mathcal{T}$ if the topology generated by $\mathcal{B}$ is the same as $\mathcal{T}$.

Lemma. Let X be a set, $\mathcal{B}$ a basis, and $\mathcal{T}$ the topology generated by $\mathcal{B}$. Then $\mathcal{T}$ is equal to the collection of all unions of elements of $\mathcal{B}$.

Proof. Since basis elements are open, unions of basis elements are open. Therefore, the collection of all unions of basis elements is contained in $\mathcal{T}$.

Conversely, let $U \in \mathcal{T}$. I must express U as a union of basis elements. For each $x \in U$, find a basis element B_x such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$. $\square$

Here are two more facts about bases. The proofs are straightforward.

1. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X generated by bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Then $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if and only if for every $U \in \mathcal{B}_2$ and $x \in U$, there is an element $V \in \mathcal{B}_1$ such that $x \in V \subset U$.

2. Given a topology $\mathcal{T}$ on X, a subcollection $\mathcal{B}$ of open sets is a basis for $\mathcal{T}$ if for all $U \in \mathcal{T}$ and $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

It's possible to go to a "lower" level and generate a topology by taking unions and finite intersections.

Definition. If X is a set, a **subbasis** is a collection of subsets of X whose union is X.

Proposition. If $\mathcal{S}$ is a subbasis in X, the collection of all unions of finite intersections of elements of $\mathcal{S}$ is a topology on X.

Proof. Let $\mathcal{T}$ denote the collection of all unions of finite intersections of elements of $\mathcal{S}$.

The union of the elements of the subbasis is X, so $X \in \mathcal{T}$. Taking the empty union gives the empty set, so $\emptyset \in \mathcal{T}$.

The union of unions of finite intersections of elements of $\mathcal{S}$ are unions of finite intersections of elements of $\mathcal{S}$, so such a union is an element of $\mathcal{T}$.

The verification that $\mathcal{T}$ is closed under finite intersections is routine, using the fact that intersections distribute over unions. I'll omit the details. $\square$

Example. (**Order topologies**) Let X be a set linearly ordered by a relation $<$. Consider the following collection of subsets of X:

1. All intervals $[s, b)$, where $b > s$ and s is the smallest element of X (if there is one).

2. All intervals $(a, l]$, where $a < l$ and l is the largest element of X (if there is one).

3. All intervals (a, b), where $a < b$.

I claim that this collection is a basis.

First, if $x \in X$, I must find an element of the basis containing x. If x is the largest element, take an interval of the form $(a, x]$; if x is the smallest element, take an interval of the form $[x, b)$. If x is neither largest nor smallest, I may find a, b such that $a < x < b$. Then $x \in (a, b)$.

To verify the second axiom for a basis, just observe that the intersection of two intervals is either empty, or an interval.

The topology generated by this basis is called the **order topology**. On $\mathbb{R}$, for instance, this topology is the standard topology. □

Example. (**Partition topologies**) Let X be a set, and let $\mathcal{P}$ be a partition of X. Then $\mathcal{P}$ is a basis: the union of the sets in the collection is X, and since the sets do not intersect, the second basis axiom is satisfied vacuously.

The topology generated by $\mathcal{P}$ is called the **partition topology** on X. The open sets are unions of partition elements.

Note that if the partition consists of X itself, the partition topology is the indiscrete topology on X. And if the partition elements are the points of X, the partition topology is the discrete topology. □

Example. (**Product Topology**) Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are topological spaces. (That is, $\mathcal{S}$ is the topology on X and $\mathcal{T}$ is the topology on Y.)

Define a basis on the product $X \times Y$ by

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{S}, V \in \mathcal{T}\}.$$

I need to verify that this is a basis.

First, if $(x, y) \in X \times Y$, find open sets $U \subset X$, $V \subset Y$ such that $x \in U \subset X$ and $y \in V \subset Y$. Then $(x, y) \in U \times V \in \mathcal{B}$.

Now let U_1 and U_2 be open in X, and let V_1 and V_2 be open in Y. Suppose that

$$(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2).$$

Note that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Find open sets $U \subset X$ and $V \subset Y$ such that $x \in U \subset U_1 \cap U_2$ and $y \in V \subset V_1 \cap V_2$. Then $U \times V \in \mathcal{B}$, and

$$(x, y) \in U \times V \subset (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Therefore, $\mathcal{B}$ is a basis.

The topology generated by $\mathcal{B}$ is the **product topology** on $X \times Y$.

In fact, if $\mathcal{C}$ is a basis for the topology on X and $\mathcal{D}$ is a basis for the topology on Y, then

$$\mathcal{E} = \{U \times V \mid U \in \mathcal{C}, V \in \mathcal{D}\}$$

is a basis for the product topology on $X \times Y$.

For instance, give $\mathbb{R}$ the standard topology generated by the open intervals. Then the product topology on $\mathbb{R}^2$ is generated by products of open intervals, i.e. open rectangles. In fact, this topology is the same as the topology generated by open balls. This will follow from the next lemma, which gives an easy way of telling when two topologies are the same. □

Lemma. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Then $\mathcal{T}_1 \subset \mathcal{T}_2$ if and only if for every $U \in \mathcal{T}_1$ and $x \in U$, there is an element $V \in \mathcal{T}_2$ such that $x \in V \subset U$.

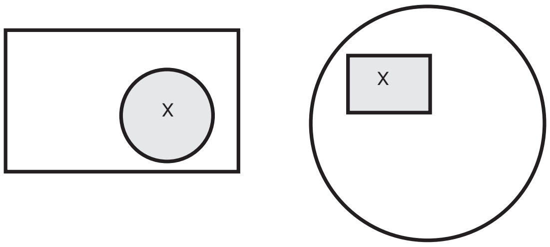
Proof. Suppose $\mathcal{T}_1 \subset \mathcal{T}_2$. If $U \in \mathcal{T}_1$ and $x \in U$, then $U \in \mathcal{T}_2$, so $x \in U \subset U$, and the condition holds.

Conversely, suppose that for every $U \in \mathcal{T}_1$ and $x \in U$, there is an element $V \in \mathcal{T}_2$ such that $x \in V \subset U$.

Let U be open in $\mathcal{T}_1$. I want to show that $U \in \mathcal{T}_2$. For each $x \in U$, find an element $V_x \in \mathcal{T}_2$ such that $x \in V_x \subset U$. Then $\bigcup_{x \in U} V_x \in \mathcal{T}_2$, but clearly $\bigcup_{x \in U} = U$. Therefore, $U \in \mathcal{T}_2$. Hence, $\mathcal{T}_1 \subset \mathcal{T}_2$. $\square$

Corollary. The product topology on $\mathbb{R}^2$ is the same as the topology generated by the open balls.

Proof. The picture below indicates the idea:



$\square$

Example. (**Subspace topology**) Let X be a topological space, and let $A \subset X$. Define a topology on A by stipulating that the open sets be the intersections of open sets of X with A.

It is easy to check that this is a topology. It is called the **subspace topology** on A. When a subset A of X is given the subspace topology, A is said to be a **subspace** of X.

It is also easy to check that if $\mathcal{B}$ is a basis for the topology on X, then

$$\{U \cap A \mid U \in \mathcal{B}\}$$

is a basis for the subspace topology on A. $\square$

Remark. If $A \subset X$, a subset of A that is open in the subspace topology need not be open in X.

For example, consider the subset $\mathbb{Z} \subset \mathbb{R}$, where $\mathbb{R}$ has the standard topology. Then $\{0\}$ is open in the subspace topology on $\mathbb{Z}$, since $(-1, 1)$ is open in $\mathbb{R}$ and $\{0\} = \mathbb{Z} \cap (-1, 1)$.

However, $\{0\}$ is not open in $\mathbb{R}$. Take $0 \in \{0\}$. There is no open interval (a, b) in $\mathbb{R}$ such that $0 \in (a, b) \subset \{0\}$.

In fact, the subspace topology on $\mathbb{Z}$ is just the discrete topology. $\square$

Theorem. Let X and Y be topological spaces. Let A be a subspace of X, and let B be a subspace of Y. The product topology on $A \times B$ agrees with topology it inherits as a subset of $X \times Y$.

Proof. Let U be open in A and let V be open in B. Thus,

$$U = A \cap U' \quad \text{and} \quad V = B \cap V',$$

where U' is open in X and V' is open in Y. $U \times V$ is a basic open set in the product topology.

Now

$$U \times V = (A \cap U') \times (B \cap V') = (A \times B) \cap (U' \times V').$$

Therefore, $U \times V$ is open in the subspace topology on $A \times B$. Thus, the product topology is contained in the subspace topology.

Going the other way, consider a basic open set $(A \times B) \cap (U' \times V')$ in the subspace topology on $A \times B$. Then

$$(A \times B) \cap (U' \times V') = (A \cap U') \times (B \cap V').$$

This is a product of an open set in A and an open set in B, so it's open in the product topology on $A \times B$.

Since an arbitrary open set in the subspace topology on $A \times B$ is a union of basic open sets, it follows that the subspace topology is contained in the product topology. $\square$