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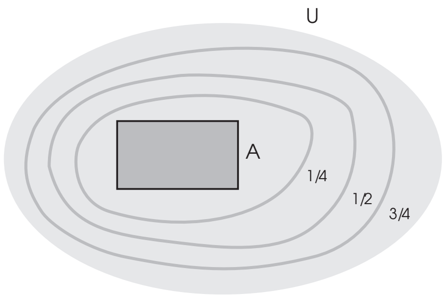
Urysohn's Lemma

Urysohn's lemma is often expressed by saying that disjoint closed sets in a normal space can be separated by a continuous function — that is, there is a continuous, real-valued function which is 0 on one of the closed sets and 1 on the other.

Note that if A and B are disjoint closed sets in a topological space X, then $X - B$ is an open set containing A. Conversely, if A is closed and U is an open set containing A, then A and $X - U$ are disjoint closed sets.

Thus, Urysohn's lemma can be expressed in another form: In a normal space, given a closed set and an open set containing it, there is a continuous, real-valued function which is 0 on the closed set and 1 outside the open set. It is this version that I'll prove; the discussion above shows that the other version is an easy corollary.

The idea is that, for such a function to exist, there should be "level sets" — a set where the function is equal to $\dfrac{1}{4}$, a set where the function is equal to $\dfrac{1}{2}$, and so on from 0 to 1. Going the other way, maybe I can define such a function by constructing the level sets.



There are various ways of constructing the level sets; they differ in the way they index the level sets. I will take the approach of [1] and [2], which index the level sets using the **dyadic rationals** in $[0, 1]$ — the rationals which can be written in the form $\dfrac{k}{2^n}$. The first step is to show that, given an appropriate collection of such level sets, one may define a *continuous* function by using the level set indices in the obvious way.

Lemma. Let X and Y be topological spaces, let $f : X \to Y$, and let $\mathcal{S}$ be a subbasis for the topology on Y. f is continuous if and only if $f^{-1}(U)$ is open in X for every $U \in \mathcal{S}$.

Proof. Since subbasic sets are open, if f is continuous, then $f^{-1}(U)$ is open in X for every $U \in \mathcal{S}$.

Conversely, suppose that $f^{-1}(U)$ is open in X for every $U \in \mathcal{S}$. An arbitrary open set in Y is a union of finite intersections of elements of $\mathcal{S}$. Since f^{-1} preserves arbitrary unions and arbitrary intersections, the inverse image of an arbitrary open set in Y is open in X. Therefore, f is continuous. □

Lemma. Let X be a topological space, and let

$$S = \left\{ \frac{k}{2^n} \;\middle|\; n \geq 0, \quad 0 \leq k \leq 2^n \right\}.$$

Suppose that for each $s \in S$, there is an open set U_s in X such that if $r, s \in S$ and $r < s$, then $\overline{U_r} \subset U_s$. The function $f : X \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \inf_{\mathbb{R}} \{ r \in S \mid x \in U_r \} & \text{if } x \in U_1 \\ 1 & \text{if } x \notin U_1 \end{cases}$$

is continuous.

Note that some elements of S are represented by more than one fraction of the form $\dfrac{k}{2^n}$ — but repetitions are eliminated by implication, since S is a *set*.

Proof. The open rays $(-\infty, a)$, $(b, +\infty)$ for $a, b \in \mathbb{R}$ form a subbasis for the standard topology on $\mathbb{R}$. It therefore suffices to show that $f^{-1}((-\infty a))$ and $f^{-1}((b, +\infty))$ are open for all $a, b \in \mathbb{R}$.

Consider the set $f^{-1}((-\infty, a))$. The range of f lies in $[0, 1]$. Thus, if $a > 1$, then $f^{-1}((-\infty, a)) = X$, and if $a \leq 0$, then $f^{-1}((-\infty, a)) = \emptyset$. In either case, $f^{-1}((-\infty, a))$ is open.

Suppose then that $0 < a \leq 1$. I will show that $f^{-1}((-\infty, a)) = \bigcup_{r<a} U_r$.

Let $x \in f^{-1}((-\infty, a))$, so $f(x) < a \leq 1$. Since $f(x)$ is the greatest lower bound of the indices r such that $x \in U_r$, and since $f(x) < a$, there is an index $r < a$ such that $x \in U_r$. Hence, $x \in \bigcup_{r<a} U_r$.

Conversely, suppose $x \in \bigcup_{r<a} U_r$. Suppose $x \in U_r$, where $r < a$. Then $f(x) \leq r < a$, so $x \in f^{-1}((-\infty, a))$.

Thus, $f^{-1}((-\infty, a)) = \bigcup_{r<a} U_r$, so $f^{-1}((-\infty, a))$ is open.

Now consider the set $f^{-1}((b, +\infty))$. The range of f lies in $(-\infty, 1]$. Thus, if $b \geq 1$, then $f^{-1}((b, +\infty)) = \emptyset$, and if $b < 0$, then $f^{-1}((b, +\infty)) = X$. In either case, $f^{-1}((b, +\infty))$ is open.

Suppose then that $0 \leq b < 1$. I will show that $f^{-1}((b, +\infty)) = \bigcup_{r>b} (X - \overline{U_r})$.

Let $x \in f^{-1}((b, +\infty))$, so $f(x) > b$. Since in addition $b < 1$, I may find $r, s \in S$ such that $b < r < s < f(x)$; by construction, $\overline{U_r} \subset U_s$. Since $f(x)$ is the greatest lower bound of the indices t such that $x \in U_t$, and since $f(x) > s$, it follows that $x \notin U_s$. Hence, $x \notin \overline{U_r}$, so $x \in X - \overline{U_r}$, and $x \in \bigcup_{r>b} (X - \overline{U_r})$.

Conversely, suppose $x \in \bigcup_{r>b} (X - \overline{U_r})$. Suppose $x \in X - \overline{U_r}$, where $r > b$. Then $f(x) \notin U_r$, so the fact that the U's are nested implies that $f(x) \notin U_s$ for all $s \in S$ with $s \leq r$. Therefore, $f(x) \geq r > b$, so $x \in f^{-1}((b, +\infty))$.

Thus, $f^{-1}((b, +\infty)) = \bigcup_{r>b} (X - \overline{U_r})$, so $f^{-1}((b, +\infty))$ is open.

Therefore, f is continuous. □

Theorem. (**Urysohn's Lemma**) Let X be a normal space, let C be a closed subset of X, and let U be an open set containing C. There is a continuous function $f : X \to [0, 1]$ such that $f(C) = 0$ and $f(X - U) = 1$.

Proof. I'll define a sequence of sets U_r indexed by

$$S = \left\{ \frac{k}{2^n} \;\middle|\; n \geq 0, \quad 0 \leq k \leq 2^n \right\}$$

and satisfying $\overline{U_r} \subset U_s$ for $r < s$.

Let $U_1 = U$. By normality, I may find an open set U_0 such that $C \subset U_0 \subset \overline{U_0} \subset U_1$.

By normality, I may find an open set $U_{1/2}$ such that

$$\overline{U_0} \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_1.$$

By normality, I may find open sets $U_{1/4}$ and $U_{3/4}$ such that

$$\overline{U_0} \subset U_{1/4} \subset \overline{U_{1/4}} \subset U_{1/2}$$

and

$$\overline{U_{1/2}} \subset U_{3/4} \subset \overline{U_{3/4}} \subset U_1.$$

Keep going. By construction, the U's satisfy the hypotheses of the lemma. The function f constructed by the lemma satisfies the conclusion of the theorem. □

Corollary. Let X be a normal space, and let A and B be disjoint closed subset of X. There is a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. □

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[1] Glen Bredon, *Topology and Geometry.*New York: Springer-Verlag New York, Inc., 1993. [ISBN 0-387-97926-3].

[2] Serge Lang, *Real Analysis* (2nd edition). Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1983. [ISBN 0-201-14179-5].