Derivations of Common PDEs

MATH 467 *Partial Differential Equations*

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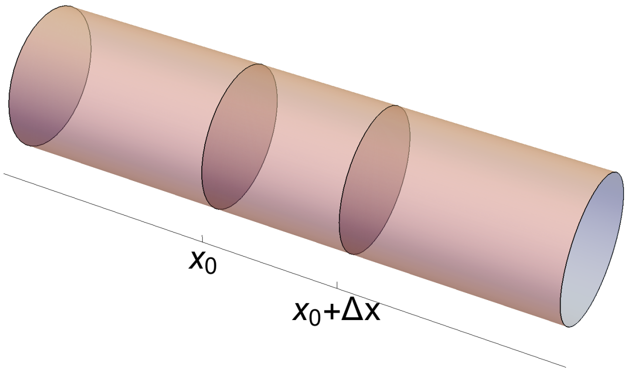
Objectives

In this lesson we will derive first- and second-order PDEs arising as mathematical models of common physical phenomena. Applications considered will include

- gas flow through a pipe,
- flow of heat energy through a rod,
- vibration of a flexible string.

Application: Gas Flow

Consider a gas flowing through a tube whose axis is parallel to the x-axis.



Assumptions

- The **density** (mass per unit volume) of the gas at position x at time t will be $\rho(x,t)$.
- The **velocity** of the gas at position x at time t will be $v(x,t)\mathbf{i}$.
- The **cross sectional area** of the tube is a constant $A > 0$.

Net Change in Mass

The mass of gas passing position x_0 at time t in an interval Δt is

$$A\,\rho(x_0, t)\,v(x_0, t)\,\Delta t.$$

The net change in mass in the interval $[x_0, x_0 + \Delta x]$ during time interval Δt is

$$\begin{aligned} & A \int_{x_0}^{x_0+\Delta x} [\rho(x, t + \Delta t) - \rho(x, t)]\; dx \\ \approx \; & -A\,\rho(x_0 + \Delta x, t) v(x_0 + \Delta x, t) \Delta t + A\,\rho(x_0, t) v(x_0, t) \Delta t. \end{aligned}$$

Divide both side of this approximation by $A\,\Delta t\,\Delta x$.

Continuity Equation

$$
\begin{aligned}
&\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x}\frac{1}{\Delta t}\left[\rho(x,t+\Delta t)-\rho(x,t)\right]\,dx \\
\approx\ & \frac{1}{\Delta x}\left(\rho(x_0,t)v(x_0,t)-\rho(x_0+\Delta x,t)v(x_0+\Delta x,t)\right)
\end{aligned}
$$

Let $\Delta t \to 0$ and $\Delta x \to 0$ to obtain

$$
\begin{aligned}
\rho_t &= -(\rho\, v)_x \\
\rho_t + (\rho\, v)_x &= 0
\end{aligned}
$$

which is called the **continuity equation.**

Solving the Continuity Equation

If $v(x,t)$ is a known function and the initial density of the gas in the tube $\rho(x,0) = \rho_0(x)$ is known then solving the continuity equation is equivalent to solving the initial value problem

$$\begin{aligned} \rho_t + v(x,t)\rho_x + v_x(x,t)\rho &= 0 \\ \rho(x,0) &= \rho_0(x). \end{aligned}$$

Note: Since no boundary conditions are specified, assume the tube is infinitely long in both directions.

Case: $v(x,t) = v_0$ a Constant

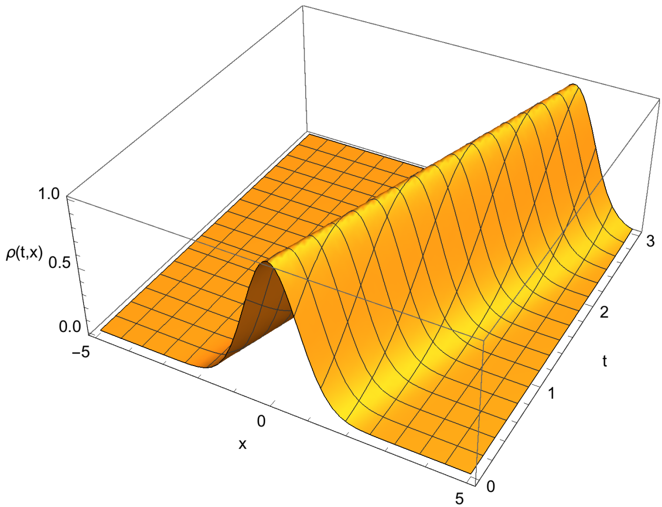
The initial value problem simplifies to

$$\begin{aligned} \rho_t + v_0 \rho_x &= 0 \\ \rho(x,0) &= \rho_0(x). \end{aligned}$$

1. Confirm that $\rho(x,t) = \rho_0(x - v_0 t)$ is a solution.
2. Interpret the solution.

Illustration

Suppose $\rho_0(x) = e^{-x^2}$ and $v_0 = 1$ then the graph of the solution $\rho(x,t) = e^{-(x-t)^2}$ resembles the following.



Case: $v(x,t) = \alpha\, x$ Where $\alpha > 0$

The initial value problem simplifies to

$$\begin{aligned} \rho_t + \alpha\, x\, \rho_x + \alpha\, \rho &= 0 \\ \rho(x,0) &= \rho_0(x). \end{aligned}$$

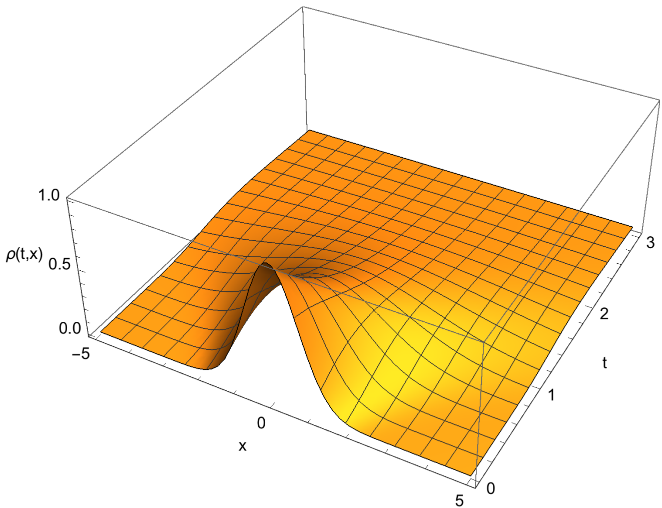
1. Confirm that $\rho(x,t) = \rho_0(x\, e^{-\alpha\, t})e^{-\alpha t}$ is a solution.
2. Interpret the solution.

Illustration

Suppose $\rho_0(x) = e^{-x^2}$ and $\alpha = 1$ then the graph of the solution

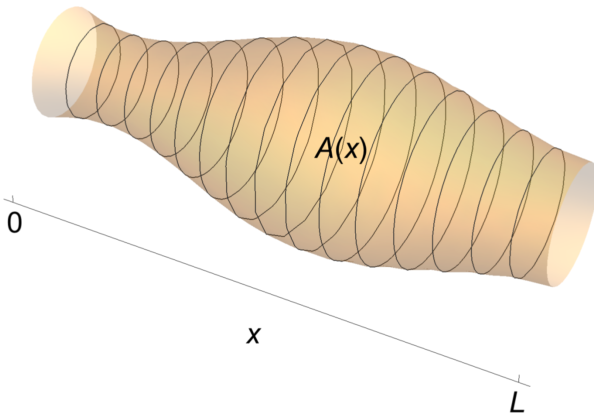
$$\rho(x,t) = e^{-(x\,e^{-t})^2} e^{-t} = e^{-t-x^2 e^{-2t}}$$

resembles the following.



Application: Heat Flow

Consider a flow of heat energy in a rod of length L whose axis is parallel to the x-axis.



Assumptions

- The sides of the rod are insulated completely. No heat exchange takes place with the surrounding media through the lateral surfaces of the rod.
- The cross sectional areas and the mass density (mass per unit length) of the rod are a function of x only, denoted as $A(x)$ and $\rho(x)$ respectively.
- Within each cross-section, the temperature is evenly distributed. This implies that the temperature distribution is a function of location t and time x only.
- The cross sectional area $A(x)$ and the density $\rho(x)$ are both smooth functions of x and the temperature throughout the rod varies smoothly in both t and x.

Definitions

$u(x,t)$: temperature of rod at time $t \geq 0$ and position $x \in [0, L]$.

$c(x)$: specific heat capacity of rod, the amount of heat energy per unit mass needed to raise the temperature of the rod by one degree,

$Q(x,t)$: heat generated per unit time per unit volume.

$\phi(x,t)$: heat flux, the energy per unit time per unit area passing location x at time t.

Partition the Rod

Consider $[a, b] \subset [0, L]$.

- Total amount of heat energy contained in $[a, b]$ at time t is

$$q(t) = \int_a^b c(x)\rho(x)A(x)u(x,t)\,dx.$$

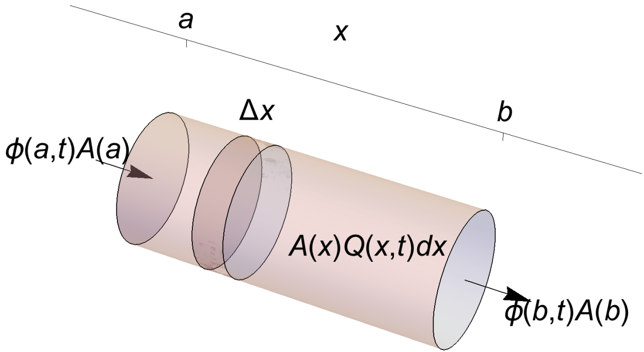
- Total amount of heat energy generated in $[a, b]$ at time t is

$$\gamma(t) = \int_a^b Q(x,t)A(x)\,dx.$$

- The net heat flux in $[a, b]$ at time t is

$$\phi(a,t)A(a) - \phi(b,t)A(b) = -\int_a^b \frac{\partial}{\partial x}\left[\phi(x,t)A(x)\right]\,dx.$$

Illustration



Conservation of Heat Energy

$$\left(\begin{array}{c}\text{Rate of change with}\\ \text{respect to time of the}\\ \text{total heat energy in}\\ \text{the interval } [a,b]\end{array}\right) = \left(\begin{array}{c}\text{Rate at which}\\ \text{heat energy}\\ \text{flows in or out}\\ \text{of the two ends}\end{array}\right) + \left(\begin{array}{c}\text{Rate at which}\\ \text{heat energy}\\ \text{is generated in}\\ \text{interval } [a,b]\end{array}\right)$$

This can be expressed mathematically as

$$\begin{aligned}
\frac{d}{dt}[q(t)] &= \phi(a,t)A(a) - \phi(b,t)A(b) + \gamma(t) \\
\frac{d}{dt}\int_a^b c(x)\rho(x)A(x)u(x,t)\,dx &= -\int_a^b \frac{\partial}{\partial x}[\phi(x,t)A(x)]\,dx \\
&\quad + \int_a^b Q(x,t)A(x)\,dx
\end{aligned}$$

Important Lemmas (1 of 2)

Lemma (Leibniz Integral Rule)

If f and $\partial f/\partial t$ are both continuous in some region of the xt-plane, including the region where $a(t) \leq x \leq b(t)$ for $t_1 \leq t \leq t_2$ and if $a(t)$ and $b(t)$ are continuously differentiable for $t_1 \leq t \leq t_2$, then

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} f(x,t)\,dx\right] = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}\left[f(x,t)\right]\,dx + f(b(t),t)\frac{db}{dt} - f(a(t),t)\frac{da}{dt}.$$

Important Lemmas (2 of 2)

Lemma

If $f(x)$ is continuous on an interval $[0, L]$, and

$$\int_a^b f(x)\,dx = 0$$

on any interval $[a, b] \subset [0, L]$, then $f(x) = 0$ for all $x \in [0, L]$.

Simplification

Assuming we can bring the derivative with respect to t inside the integral the previous equation may be re-written as $\int_a^b \left(c(x)\rho(x)A(x)\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\left(\phi(x,t)A(x)\right) - Q(x,t)A(x) \right)\, dx = 0.$ Since the interval $[a,b] \subset [0,L]$ is arbitrary then the integrand must be 0.

$$c(x)\rho(x)A(x)\frac{\partial}{\partial t}\left[u(x,t)\right] = -\frac{\partial}{\partial x}\left[\phi(x,t)A(x)\right] + Q(x,t)A(x)$$

for $t \geq 0$ and $x \in [0,L]$.

Fourier's Law

$$c(x)\rho(x)A(x)\frac{\partial}{\partial t}[u(x,t)] = -\frac{\partial}{\partial x}[\phi(x,t)A(x)] + Q(x,t)A(x)$$

This equation contains two unknowns $u(x,t)$ and $\phi(x,t)$.

Fourier's Law of Heat Conduction: heat energy flows from warmer regions to cooler regions at a rate proportional to the difference in the temperatures between the regions.

$$\phi(x,t) = -K_0(x)\frac{\partial u}{\partial x}(x,t)$$

Function $K_0(x) \geq 0$ is called the **thermal conductivity** and measures the ability of the rod to conduct heat energy.

Simplification

Using Fourier's Law the conservation of heat energy equation becomes

$$c(x)\rho(x)A(x)\frac{\partial}{\partial t}[u(x,t)] = -\frac{\partial}{\partial x}\left[-K_0(x)\frac{\partial u}{\partial x}(x,t)A(x)\right]+Q(x,t)A(x).$$

- If c, ρ, A, and K_0 are constant then

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{1}{c\rho}Q(x,t)$$

  where $k = K_0/(c\,\rho)$ is called the **thermal diffusivity.**
- In addition if $Q(x,t) \equiv 0$ then

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

  which referred to as the **one-dimensional heat equation.**

IBVPs for the Heat Equation

- An **initial condition** specifies the distribution of temperature in the rod at $t = 0$.

$$u(x, 0) = f(x) \quad \text{for } 0 \leq x \leq L$$

- There are three different classes of boundary conditions which can be specified.

Dirichlet Boundary Conditions

If the temperatures at the two ends of the rod are specified for $t > 0$ as

$$\begin{aligned} u(0,t) &= g_1(t) \\ u(L,t) &= g_2(t). \end{aligned}$$

These are known as **Dirichlet boundary conditions** or **boundary conditions of the first kind**.

The simplest such BCs are homogeneous:

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0. \end{aligned}$$

Question: what physical interpretation can be given to these BCs?

Neumann Boundary Conditions

If the heat fluxes at the two ends of the rod are specified for $t > 0$ as

$$\begin{aligned} -K_0 u_x(0,t) &= g_1(t) \\ K_0 u_x(L,t) &= g_2(t). \end{aligned}$$

These are known as **Neumann boundary conditions** or **boundary conditions of the second kind.**

The simplest such BCs are homogeneous:

$$\begin{aligned} u_x(0,t) &= 0 \\ u_x(L,t) &= 0. \end{aligned}$$

Question: what physical interpretation can be given to these BCs?

Robin Boundary Conditions

If the rate of heat loss (or gain) between the two ends of the rod and the surrounding media is proportional to the differences in the temperatures at the two ends and the surrounding media for $t > 0$ then

$$\begin{aligned} K_0 u_x(0,t) &= \alpha(u(0,t) - g_1(t)) \\ K_0 u_x(L,t) &= -\beta(u(L,t) - g_2(t)). \end{aligned}$$

These are known as **Robin boundary conditions** or **boundary conditions of the third kind.**

- The constants $\alpha > 0$ and $\beta > 0$ are called the **coefficients of convective heat transfer.**
- Each boundary condition is equivalent to **Newton's Law of Cooling**.
- The Dirichlet and Neumann BCs are limiting cases of boundary conditions of the third kind.

Example

Suppose a one-dimensional rod of length L is subject to a constant heat flux ϕ_0 at $x = 0$ and constant temperature u_0 at $x = L$. The initial temperature distribution of the rod is $f(x)$.

1. Set up the IBVP modeling this situation.
2. Solve for the steady-state temperature distribution in the rod.

Application: Vibrating String

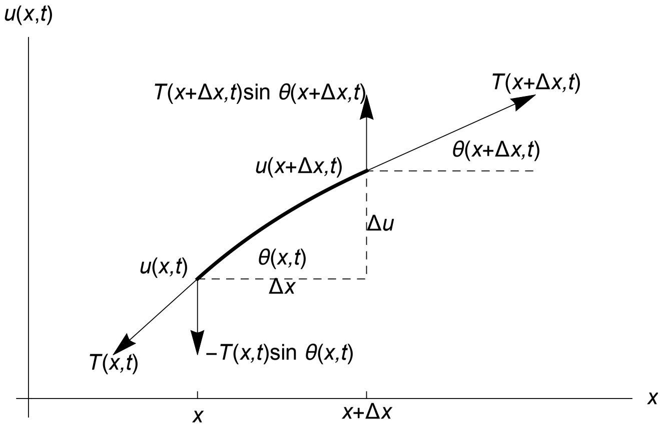
- Consider a flexible string in the xu-plane whose horizontal coordinates lie in the interval $[0, L]$.
- Let $u(x, t)$ denote the displacement of the string from equilibrium at time t and position x.
- Displacement of the string is confined to the xu-plane.

Assumptions

- The displacement $u(x, t)$ of the string at all points is small.
- The string is perfectly flexible and offers no resistance to bending.
- The density (mass per unit length) of the string is $\rho(x)$.
- The string is under tension (a force) whose magnitude is $T(x, t)$. The tension acts in the direction tangent to the string at point $(x, u(x, t))$.
- The angle between the tangent line to the string at each point and the positive x-axis will be $\theta(x, t)$. These angles will be assumed small.
- The vertical component (parallel to the direction of displacement) of all the external forces per unit length acting on the string is $f(x, t)$.

Illustration

Consider a small segment of the string with endpoints at $(x, u(x,t))$ and $(x+\Delta x, u(x+\Delta x, t))$.



Newton's Second Law of Motion

- Mass of string element: $\rho(x)\sqrt{(\Delta x)^2 + (\Delta u)^2}$.
- Vertical component of tension at $(x, u(x,t))$: $T(x,t)\sin\theta(x,t)$.
- Vertical component of tension at $(x + \Delta x, u(x + \Delta x, t))$: $T(x + \Delta x, t)\sin\theta(x + \Delta x, t)$.

According to Newton's Second Law of Motion,

$$\begin{aligned} m\,a &= \sum F \\ \rho(x)\sqrt{(\Delta x)^2 + (\Delta u)^2}\frac{\partial^2 u}{\partial t^2} &= T(x + \Delta x, t)\sin\theta(x + \Delta x, t) \\ &\quad - T(x,t)\sin\theta(x,t) + f(x,t)\Delta x. \end{aligned}$$

Divide both sides by Δx.

Simplification

$$\rho(x)\sqrt{1+\left(\frac{\Delta u}{\Delta x}\right)^2}\frac{\partial^2 u}{\partial t^2} = \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x} + f(x,t)$$

Let $\Delta x \to 0$.

$$\begin{aligned}\rho(x)\sqrt{1+(u_x)^2}\,u_{tt} &= \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x,t)\right] + f(x,t)\\ &= T_x \sin\theta(x,t) + T(x,t)\theta_x\cos\theta(x,t) + f(x,t)\end{aligned}$$

Trigonometric Relationships

The following relationships will simplify the last equation further.

$$\begin{aligned}
\tan\theta(x,t) &= u_x \\
\sin\theta(x,t) &= \frac{u_x}{\sqrt{1+(u_x)^2}} \\
\cos\theta(x,t) &= \frac{1}{\sqrt{1+(u_x)^2}}
\end{aligned}$$

Assuming $\theta(x,t)$ is small implies u_x is small and

$$\begin{aligned}
\sqrt{1+(u_x)^2} &\approx 1 \\
\sin\theta(x,t) &\approx u_x \\
\cos\theta(x,t) &\approx 1.
\end{aligned}$$

Simplification

Using these trigonometric relationships simplifies the previous result to the form

$$\rho(x)u_{tt} = T_x u_x + T(x,t)\theta_x + f(x,t).$$

Remarks:

- this equation involves three unknowns: $u(x,t)$, $T(x,t)$, and $\theta(x,t)$.
- using the chain rule

$$u_{xx} = \frac{\partial}{\partial\theta}\left[u_x\right]\theta_x = \frac{\partial}{\partial\theta}\left[\tan\theta\right]\theta_x = (\sec^2\theta)\theta_x \approx \theta_x$$

- substituting into the equation above eliminates θ_x.

$$\rho(x)u_{tt} = T_x u_x + T(x,t)u_{xx} + f(x,t).$$

Eliminating the Tension

The horizontal components of the tension at $(x, u(x,t))$ and $(x + \Delta x, u(x + \Delta x, t))$ cancel each other since there is no acceleration parallel to the x-axis.

$$\begin{aligned}
T(x+\Delta x, t)\cos\theta(x+\Delta x, t) - T(x,t)\cos\theta(x,t) &= 0 \\
\lim_{\Delta x \to 0} \frac{T(x+\Delta x, t)\cos\theta(x+\Delta x, t) - T(x,t)\cos\theta(x,t)}{\Delta x} &= 0 \\
\frac{\partial}{\partial x}\left[T(x,t)\cos\theta(x,t)\right] &= 0 \\
T_x \cos\theta(x,t) - T(x,t)(\sin\theta(x,t))\theta_x &= 0 \\
T(x,t) u_x u_{xx} &\approx T_x
\end{aligned}$$

Thus the previous equation for the string simplifies to

$$\rho(x) u_{tt} = T(x,t)\,(u_x)^2\, u_{xx} + T(x,t) u_{xx} + f(x,t).$$

Simplification

Using the assumption that u_x is small, then

$$\rho(x)u_{tt} = T(x,t)u_{xx} + f(x,t).$$

If the tension and density of the string are constants then

$$\begin{aligned}
\rho_0 u_{tt} &= T_0 u_{xx} + f(x,t) \\
u_{tt} &= \frac{T_0}{\rho_0} u_{xx} + \frac{1}{\rho_0} f(x,t) \\
u_{tt} &= c^2 u_{xx} + \frac{1}{\rho_0} f(x,t).
\end{aligned}$$

If there are no external forces then

$$u_{tt} = c^2 u_{xx},$$

which is called the **one-dimensional wave equation.**
Question: what are the units of constant c?

IBVPs for the Wave Equation

- An **initial condition** specifies the **displacement** and **velocity** of the string at $t = 0$.

$$\begin{aligned} u(x,0) &= f(x) \quad \text{for } 0 \le x \le L \\ u_t(x,0) &= g(x) \end{aligned}$$

- There are three different classes of boundary conditions which can be specified.

Dirichlet Boundary Conditions

If the displacements of the two ends of the string are specified for $t > 0$ as

$$\begin{aligned} u(0,t) &= f_1(t) \\ u(L,t) &= f_2(t). \end{aligned}$$

These are known as **Dirichlet boundary conditions** or **boundary conditions of the first kind**.

The simplest such BCs are homogeneous:

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0. \end{aligned}$$

Question: what physical interpretation can be given to these BCs?

Neumann Boundary Conditions

If the slopes of the string at the two ends are specified for $t > 0$ as

$$\begin{aligned} T_0 u_x(0,t) &= g_1(t) \\ T_0 u_x(L,t) &= g_2(t). \end{aligned}$$

These are known as **Neumann boundary conditions** or **boundary conditions of the second kind.**

The simplest such BCs are homogeneous:

$$\begin{aligned} u_x(0,t) &= 0 \\ u_x(L,t) &= 0. \end{aligned}$$

Question: what physical interpretation can be given to these BCs?

Robin Boundary Conditions

Robin boundary conditions or **boundary conditions of the third kind** take the form of

$$\begin{aligned} T_0 u_x(0,t) &= k_1(u(0,t) - d_1(t)) \\ T_0 u_x(L,t) &= -k_2(u(L,t) - d_2(t)). \end{aligned}$$

If $d_1(t) \equiv 0$ and $d_2(t) \equiv 0$ the following homogeneous boundary conditions arise.

$$\begin{aligned} T_0 u_x(0,t) &= k_1 u(0,t) \\ T_0 u_x(L,t) &= -k_2 u(L,t). \end{aligned}$$

The Dirichlet ($T_0 = 0$) and Neumann ($k_1 = k_2 = 0$) boundary conditions are special cases of these BCs.

Homework

- Read Sections 1.3 and 1.4
- Exercises: 7–11