

# Preliminary Definitions and Examples

MATH 467 *Partial Differential Equations*

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# What is a PDE?

## Definition

A **partial differential equation** (PDE) is an equation involving an unknown function  $u$ , of two or more independent variables  $t$ ,  $x$ ,  $y$ , *etc* and the partial derivatives of  $u$ .

# Examples

$$u_t + u u_y = 0$$

$$u_{xx} + u_{yy} = f(x, y)$$

$$u_{xx} + x y^3 u_{yy} - e^x u_z + x y u = 0$$

$$u_t - u_{xx} + (u_x)^2 = 0$$

$$u_t + c u u_x + u_{xxx} = 0$$

# Partial Derivative Notation

$$\begin{array}{rcl} u_t & = & \frac{\partial u}{\partial t} \\ u_{xx} & = & \frac{\partial^2 u}{\partial x^2} \\ u_{xy} & = & \frac{\partial^2 u}{\partial y \partial x} \\ & & \vdots \end{array}$$

# Description of PDEs

PDEs can be described as being

- ▶ linear or nonlinear (or in some cases semilinear or quasilinear),
- ▶ homogeneous or nonhomogeneous,
- ▶ first order, second order, third order, *etc.*

# Order

## Definition

The **order** of a PDE is the highest order partial derivative present in the PDE.

Find the order of each of the following PDEs.

$$u_t + u u_y = 0$$

$$u_{xx} + u_{yy} = f(x, y)$$

$$u_{xx} + x y^3 u_{yy} - e^x u_z + x y u = 0$$

$$u_t - u_{xx} + (u_x)^2 = 0$$

$$u_t + c u u_x + u_{xxx} = 0$$

# Linearity

## Definition

A PDE is **linear** if all the terms of the PDE are linear in the unknown and its partial derivatives. Otherwise the PDE is **nonlinear**.

Determine which of the following PDEs are linear or nonlinear.

$$u_t + u u_y = 0$$

$$u_{xx} + u_{yy} = f(x, y)$$

$$u_{xx} + x y^3 u_{yy} - e^x u_z + x y u = 0$$

$$u_t - u_{xx} + (u_x)^2 = 0$$

$$u_t + c u u_x + u_{xxx} = 0$$

# Solution to a PDE

## Definition

A **solution** to a PDE is a function defined on an appropriate domain which makes the PDE a true equation when substituted for the unknown of the PDE.

Consider the second-order, linear PDE

$$u_{xy} = 0 \quad \text{for } (x, y) \in \mathbb{R}^2.$$

Verify that  $u(x, y) = F(x) + G(y)$  is a solution to the PDE where  $F$  and  $G$  are arbitrary differentiable functions defined for all real numbers.

# Common Generic PDEs

We will primarily focus on properties and solutions of the following two types of PDE.

First-order Linear:

$$a(x, t)u_x + b(x, t)u_t + c(x, t)u = g(x, t)$$

Second-order Linear:

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G$$

where  $A, B, C, D, E, F$ , and  $G$  are functions of  $(x, t)$ .

**Note:** a linear PDE is called **homogeneous** if  $g(x, t) \equiv 0$  or  $G(x, t) \equiv 0$ , otherwise it is **nonhomogeneous**.

# Principle of Superposition

## Theorem

*If  $u_i$  is a solution of the second-order, linear partial differential equation*

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G_i$$

*for  $i = 1, 2, \dots, n$  on a domain  $\Omega \subset \mathbb{R}^2$ , then for any constants  $c_1, c_2, \dots, c_n$  the function  $u = c_1u_1 + c_2u_2 + \dots + c_nu_n$  is a solution of*

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = \sum_{i=1}^n c_i G_i$$

*on  $\Omega$ .*

# Principle of Subtraction

## Theorem

*If  $u_1(x, t)$  and  $u_2(x, t)$  are solutions to*

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G$$

*then  $u(x, t) \equiv u_1(x, t) - u_2(x, t)$  is a solution to*

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = 0.$$

**Remark:** any two solutions to a nonhomogeneous PDE differ by a solution to the associated homogeneous PDE.

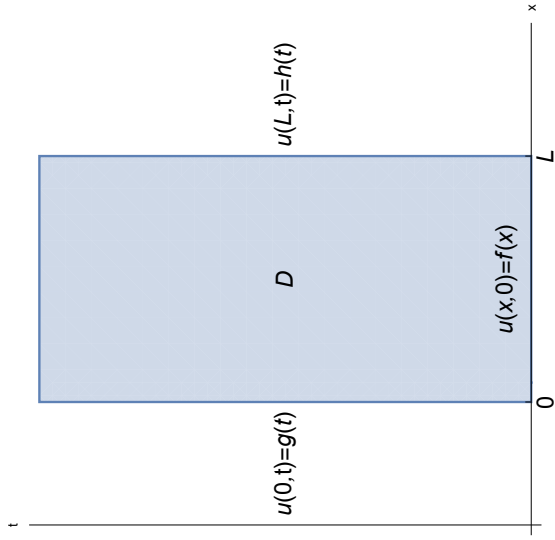
# Initial and Boundary Conditions

- ▶ If  $u(x, t)$  is the solution to a PDE which satisfies the requirement that  $u(x, 0) = f(x)$ , then this requirement is called an **initial condition**.
- ▶ If the domain of the solution  $u(x, t)$  is  $D = \{(x, t) : t \geq 0, 0 \leq x \leq L\}$  and

$$\begin{aligned}u(0, t) &= g(t) \\ u(L, t) &= h(t),\end{aligned}$$

then these requirements are called **boundary conditions**.

# Illustration



# Initial Boundary Value Problems

## Definition

A PDE along with specified initial and boundary conditions is referred to as an **initial boundary value problem** (IBVP). If only boundary conditions are specified (perhaps because  $u$  is independent of  $t$ ) the PDE is called a **boundary value problem** (BVP).

## Definition

An IBVP is **well-posed** if it has a unique solution and that solution depends continuously on the initial and boundary conditions. Otherwise the PDE is **ill-posed**.

## Example

Consider the PDE:

$$u_{xx} + u_{yy} = 0.$$

Verify that the following functions solve the PDE.

$$\begin{aligned}u_1(x, y) &= x + y \\u_2(x, y) &= x^2 - y^2 \\u_3(x, y) &= e^x \cos y \\u_4(x, y) &= \ln(x^2 + y^2) \\u_5(x, y) &= \frac{x}{x^2 + y^2}\end{aligned}$$

## Example: BVP

Consider the BVP:

$$\begin{aligned}u_{xx} + u_{yy} &= 0 \\ u(x, y) &= 0 \quad \text{for } x^2 + y^2 = 1.\end{aligned}$$

Find its solution.

## Example: IBVP

Consider the IBVP:

$$\begin{aligned}u_t - u_{xx} &= 0 && \text{for } t \geq 0, 0 \leq x \leq \pi \\u(x, 0) &= 100 + \sin x && \text{(IC)} \\u(0, t) &= 100 && \text{(BC for } x = 0) \\u(\pi, t) &= 100 && \text{(BC for } x = \pi)\end{aligned}$$

1. Show that  $u_1(x, t) = t + x^2/2$  solves the PDE.
2. Show that  $u_2(x, t) = A + Be^{-t} \sin x$  solves the PDE.
3. Choose  $A$  and  $B$  appropriately so that  $u_2(x, t)$  solves the IBVP.

# Classification of PDEs

Recall the general second-order PDE:

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G$$

where  $A, B, C, D, E, F$ , and  $G$  are functions of  $(x, t) \in D$ . The PDE is said to be

**elliptic** on  $D$  if  $4AC - B^2 > 0$  for all  $(x, t) \in D$ ,

**parabolic** on  $D$  if  $4AC - B^2 = 0$  for all  $(x, t) \in D$ ,

**hyperbolic** on  $D$  if  $4AC - B^2 < 0$  for all  $(x, t) \in D$ .

# Examples

Classify each of the following PDEs as either **elliptic**, **parabolic**, or **hyperbolic**.

$$\begin{array}{rclcl} u_{tt} + u_{xx} + u_t - 2u_x & = & f(x, t) \\ u_{tt} - u_{tx} + 2u_{xx} + e^x u_t - \sin(tx) u_x & = & f(x, t) \\ u_{tt} - 2u_{tx} + u_{xx} + 2u_t - u_x & = & 0 \\ u_{tt} + t u_{xx} & = & 0 \\ u_{xx} - \beta u_t + \alpha u_x + \rho u + f(x, t) & = & u_t \end{array}$$

# Homework

- ▶ Read Sections 1.1 and 1.2
- ▶ Exercises: 1–6