Preliminaries: Notation, Definitions, and the Principle of Superposition

Partial Differential Equations

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Fall 2025

What is a PDE?

Definition

A **partial differential equation** (PDE) is an equation involving an unknown function u, of two or more independent variables t, x, y, *etc* and the partial derivatives of u.

Examples

$$
\begin{aligned}
u_t + u\,u_y &= 0 \\
u_{xx} + u_{yy} &= f(x,y) \\
u_{xx} + x\,y^3 u_{yy} - e^x u_z + x\,y\,u &= 0 \\
u_t - u_{xx} + (u_x)^2 &= 0 \\
u_t + c\,u\,u_x + u_{xxx} &= 0
\end{aligned}
$$

Partial Derivative Notation

$$u_t = \frac{\partial u}{\partial t}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y \, \partial x}$$

$$\vdots$$

Description of PDEs

PDEs can be described as being

- linear or nonlinear (or in some cases ¯semilinear or quasilinear),
- homogeneous or nonhomogeneous,
- first order, second order, third order, *etc*.

Order

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Find the order of each of the following PDEs.

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A PDE is **linear** if all the terms of the PDE are linear in the unknown and its partial derivatives. Otherwise the PDE is **nonlinear**.

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Determine which of the following PDEs are linear or nonlinear.

$$
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Solution to a PDE

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A **solution** to a PDE is a function defined on an appropriate domain which makes the PDE a true equation when substituted for the unknown of the PDE.

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Consider the second-order, linear PDE

$$u_{xy} = 0 \quad \text{for } (x, y) \in \mathbb{R}^2.$$

Verify that $u(x, y) = F(x) + G(y)$ is a solution to the PDE where F and G are arbitrary differentiable functions defined for all real numbers.

Common Generic PDEs

We will primarily focus on properties and solutions of the following two types of PDE.

First-order Linear:

$$a(x,t)u_x + b(x,t)u_t + c(x,t)u = g(x,t)$$

Second-order Linear:

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G$$

where A, B, C, D, E, F, and G are functions of (x,t).

Note: a linear PDE is called **homogeneous** if $g(x,t) \equiv 0$ or $G(x,t) \equiv 0$, otherwise it is **nonhomogeneous**.

Principle of Superposition

Theorem

If u_i is a solution of the second-order, linear partial differential equation

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G_i$$

for $i = 1, 2, \dots, n$ on a domain $\Omega \subset \mathbb{R}^2$, then for any constants $c_1, c_2, \dots, c_n$ the function $u = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$ is a solution of

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = \sum_{i=1}^{n} c_i G_i$$

on Ω.

Principle of Subtraction

Theorem

If $u_1(x,t)$ and $u_2(x,t)$ are solutions to

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G$$

then $u(x,t) \equiv u_1(x,t) - u_2(x,t)$ is a solution to

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = 0.$$

Remark: any two solutions to a nonhomogeneous PDE differ by a solution to the associated homogeneous PDE.

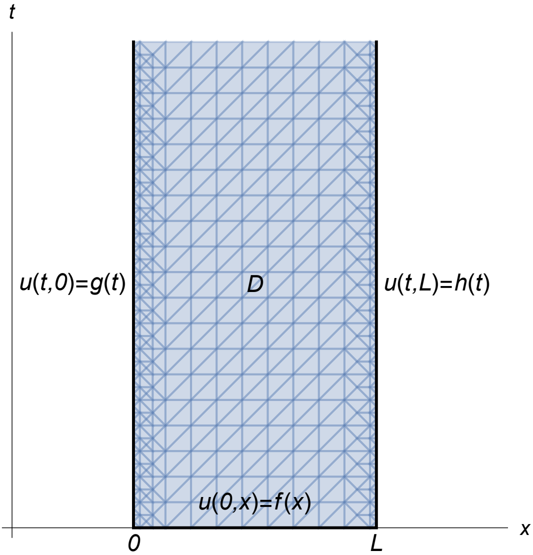
Initial and Boundary Conditions

- If $u(x,t)$ is the solution to a PDE which satisfies the requirement that $u(x,0) = f(x)$, then this requirement is called an **initial condition**.
- If the domain of the solution $u(x,t)$ is $D = \{(x,t) \,:\, t \geq 0,\, 0 \leq x \leq L\}$ and

  $$u(0,t) = g(t)$$
  $$u(L,t) = h(t),$$

  then these requirements are called **boundary conditions.**

Illustration



Initial Boundary Value Problems

Definition

A PDE along with specified initial and boundary conditions is referred to as an **initial boundary value problem** (IBVP). If only boundary conditions are specified (perhaps because u is independent of t) the PDE is called a **boundary value problem** (BVP).

Definition

An IBVP is **well-posed** if it has a unique solution and that solution depends continuously on the initial and boundary conditions. Otherwise the PDE is **ill-posed**.

Example

Consider the PDE:

$$u_{xx} + u_{yy} = 0.$$

Verify that the following functions solve the PDE.

$$\begin{aligned}
u_1(x, y) &= x + y \\
u_2(x, y) &= x^2 - y^2 \\
u_3(x, y) &= e^x \cos y \\
u_4(x, y) &= \ln(x^2 + y^2) \\
u_5(x, y) &= \frac{x}{x^2 + y^2}
\end{aligned}$$

Example: BVP

Consider the BVP:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x,y) &= 0 \quad \text{for } x^2 + y^2 = 1. \end{aligned}$$

Find its solution.

Example: IBVP

Consider the IBVP:

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{for } t \geq 0,\ 0 \leq x \leq \pi \\ u(x,0) &= 100 + \sin x \quad \text{(IC)} \\ u(0,t) &= 100 \quad \text{(BC for } x = 0\text{)} \\ u(\pi,t) &= 100 \quad \text{(BC for } x = \pi\text{)} \end{aligned}$$

1. Show that $u_1(x,t) = t + x^2/2$ solves the PDE.

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2. Show that $u_2(x,t) = A + Be^{-t}\sin x$ solves the PDE.

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1. Show that $u_1(x,t) = t + x^2/2$ solves the PDE.
2. Show that $u_2(x,t) = A + Be^{-t}\sin x$ solves the PDE.
3. Choose A and B appropriately so that $u_2(x,t)$ solves the IBVP.

Classification of PDEs

Recall the general second-order PDE:

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G$$

where A, B, C, D, E, F, and G are functions of $(x, t) \in D$. The PDE is said to be

elliptic on D if $4AC - B^2 > 0$ for all $(x, t) \in D$,
parabolic on D if $4AC - B^2 = 0$ for all $(x, t) \in D$,
hyperbolic on D if $4AC - B^2 < 0$ for all $(x, t) \in D$.

Examples

Classify each of the following PDEs as either **elliptic**, **parabolic**, or **hyperbolic**.

$$
\begin{aligned}
u_{tt} + u_{xx} + u_t - 2u_x &= f(x,t) \\
u_{tt} - u_{tx} + 2u_{xx} + e^x u_t - \sin(t\,x)u_x &= f(x,t) \\
u_{tt} - 2u_{tx} + u_{xx} + 2u_t - u_x &= 0 \\
u_{tt} + t\,u_{xx} &= 0 \\
u_{xx} - \beta\,u_t + \alpha\,u_x + \rho\,u + f(x,t) &= u_t
\end{aligned}
$$