

Separation of Variables

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn the approach of a fundamental technique for solving many PDEs, namely **separation of variables**.

This technique reduces the problem of finding the unknown dependent variable of the PDE u , which depends on n independent variables, to the problem of solving n ordinary differential equations each depending on a single independent variable.

We will assume the dependent variable is a **product solution**.

$$u(x, y, t) = X(x)Y(y)T(t)$$

Example

Apply the method of separation of variables to the equation

$$x^2 u_{xx} - 2y u_y = 0$$

and find a corresponding set of ordinary differential equations.

Solution (1 of 2)

- ▶ Assume $u(x, y) = X(x)Y(y)$ then

$$u_{xx} = X''(x)Y(y)$$

$$u_y = X(x)Y'(y).$$

- ▶ Substitute into the PDE.

$$x^2 u_{xx} - 2y u_y = 0$$

$$x^2 X''(x)Y(y) - 2yX(x)Y'(y) = 0$$

- ▶ Divide both sides by $u(x, y) = X(x)Y(y)$.

$$\frac{x^2 X''(x)Y(y)}{X(x)Y(y)} - \frac{2yX(x)Y'(y)}{X(x)Y(y)} = 0$$

$$x^2 \frac{X''(x)}{X(x)} = 2y \frac{Y'(y)}{Y(y)}$$

Solution (2 of 2)

$$x^2 \frac{X''(x)}{X(x)} = 2y \frac{Y'(y)}{Y(y)}$$

Key observation: the left-hand side is a function of x while the right-hand side is a function of y . Since they are equal they must be constant.

$$x^2 \frac{X''(x)}{X(x)} = c = 2y \frac{Y'(y)}{Y(y)}$$

This implies

$$\begin{array}{lcl} x^2 \frac{X''(x)}{X(x)} & = & c \\ 2y \frac{Y'(y)}{Y(y)} & = & c \end{array} \iff \begin{array}{lcl} x^2 X''(x) - c X(x) & = & 0 \\ 2y Y'(y) - c Y(y) & = & 0 \end{array}$$

Example

Apply the method of separation of variables to the following equation and determine the corresponding set of ordinary differential equations.

$$u_{xx} + u_x + 2u_y - u \sin x = 0$$

Example

Determine if the method of separation of variables can be applied to the following partial differential equation. If so, determine the resulting ordinary differential equations.

$$u_x + (x + y)u_y = 0.$$

Example

Determine if the method of separation of variables can be applied to the partial differential equation below. If so, determine the ordinary differential equations in each variable which result.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad (\text{for } r > 0)$$

The dependent variable is bounded as $r \rightarrow 0$ and is 2π -periodic in θ .

Example: The Heat Equation

Consider the one-dimensional, homogeneous heat equation with Dirichlet boundary conditions and an initial condition as below.

$$\begin{aligned}u_t &= k u_{xx}, & 0 < x < L, & \quad t > 0 \\u(0, t) &= 0, & t > 0 \\u(L, t) &= 0, & t > 0 \\u(x, 0) &= f(x), & 0 \leq x \leq L\end{aligned}$$

Apply the method of separation of variables to this initial boundary value problem and determine the product solutions which satisfy the homogeneous boundary conditions.

Solution (1 of 8)

- ▶ Assume the product solution $u(x, t) = X(x)T(t)$.
- ▶ Differentiating and substituting into the heat equation yields

$$\begin{aligned}u_t &= k u_{xx} \\X(x)T'(t) &= k X''(x)T(t) \\\frac{1}{k} \frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = -c\end{aligned}$$

where c is a constant and the minus sign is introduced for convenience.

- ▶ The resulting ODEs for x and t are

$$\begin{aligned}X''(x) + c X(x) &= 0 \\T'(t) + c k T(t) &= 0.\end{aligned}$$

Solution (2 of 8)

Consider the ODE for x and the boundary conditions.

$$\begin{aligned}X''(x) + c X(x) &= 0 \\u(0, t) = X(0)T(t) &= 0 \iff X(0) = 0 \\u(L, t) = X(L)T(t) &= 0 \iff X(L) = 0\end{aligned}$$

Find solutions to the ODE which satisfy the boundary conditions.
Consider the three cases:

- ▶ $c = 0$,
- ▶ $c < 0$,
- ▶ $c > 0$.

Solution (3 of 8)

Case: $c = 0$.

$$\begin{aligned}X''(x) + c X(x) &= X''(x) = 0 \\X(x) &= Ax + B\end{aligned}$$

When $x = 0$ we have $0 = X(0) = B$.

When $x = L$ we have $0 = X(L) = AL$ which implies $A = 0$. Thus when $c = 0$ we have only the trivial solution $X(x) = 0$.

Solution (4 of 8)

Case: $c < 0$. Let $c = -\lambda^2$ where $\lambda > 0$.

$$X''(x) + c X(x) = X''(x) - \lambda^2 X(x) = 0$$

$$X(x) = A \cosh(\lambda x) + B \sinh(\lambda x)$$

When $x = 0$ we have $0 = X(0) = A$.

When $x = L$ we have $0 = X(L) = B \sinh(\lambda L)$ which implies $B = 0$.
Thus when $c < 0$ we have only the trivial solution $X(x) = 0$.

Solution (5 of 8)

Case: $c > 0$. Let $c = \lambda^2$ where $\lambda > 0$.

$$X''(x) + c X(x) = X''(x) + \lambda^2 X(x) = 0$$

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x)$$

When $x = 0$ we have $0 = X(0) = A$.

When $x = L$ we have $0 = X(L) = B \sin(\lambda L)$ which implies

$$\lambda L = n\pi$$

$$\lambda \equiv \lambda_n = \frac{n\pi}{L}$$

for $n \in \mathbb{N}$. Thus when $c = n^2 \pi^2 / L^2$ for $n \in \mathbb{N}$ we have the nontrivial solutions

$$X_n(x) = \sin \frac{n\pi x}{L}.$$

Function $X_n(x)$ is called an **eigenfunction** corresponding to **eigenvalue** $n^2 \pi^2 / L^2$.

Solution (6 of 8)

Using the known eigenvalue in the ODE for t yields

$$T'(t) + \frac{n^2 \pi^2 k}{L^2} T(t) = 0$$
$$T(t) \equiv T_n(t) = C_n e^{-n^2 \pi^2 k t / L^2}$$

for $n \in \mathbb{N}$.

The product solutions which satisfy the boundary conditions have the form

$$u_n(x, t) = X_n(x) T_n(t) = B_n e^{-n^2 \pi^2 k t / L^2} \sin\left(\frac{n\pi x}{L}\right).$$

These are called **fundamental solutions**.

Solution (7 of 8)

Using the Principle of Superposition, a finite linear combination of fundamental solutions will likewise solve the PDE and satisfy the BCs.

$$u(x, t) = \sum_{n=1}^N B_n e^{-n^2 \pi^2 k t / L^2} \sin\left(\frac{n\pi x}{L}\right)$$

Now consider the initial condition:

$$u(x, 0) = f(x)$$
$$\sum_{n=1}^N B_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

As long as $f(x)$ contains a finite sum of sine functions of the appropriate periods we can equate coefficients and solve for the B_n 's.

Solution (8 of 8)

Take the case in which

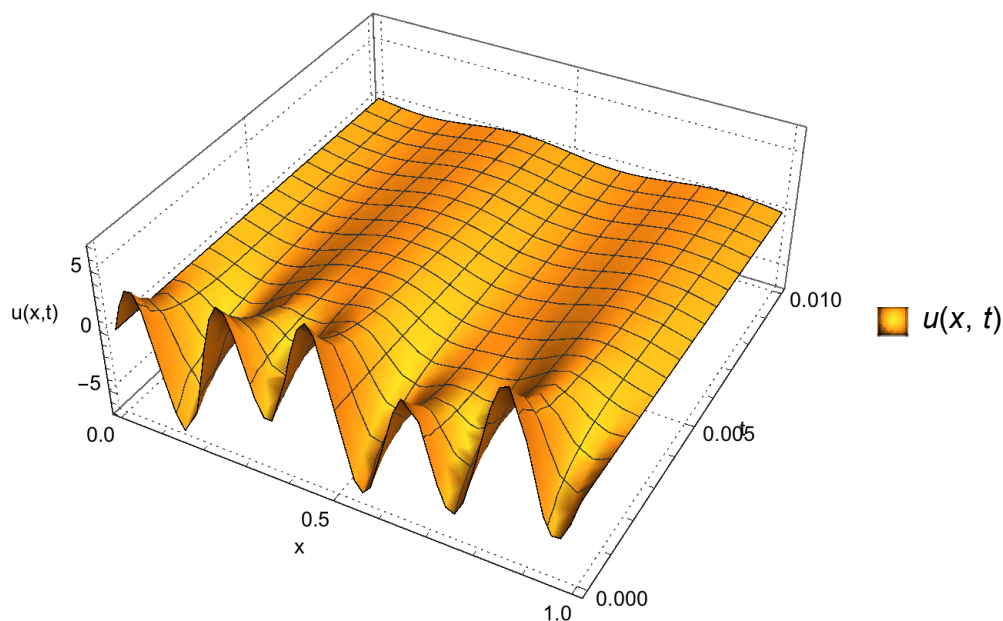
$$f(x) = -2 \sin \frac{4\pi x}{L} + 5 \sin \frac{10\pi x}{L}.$$

Then $B_4 = -2$, $B_{10} = 5$ and all other coefficients are 0.

$$u(x, t) = -2e^{-16\pi^2 k t/L^2} \sin \frac{4\pi x}{L} + 5e^{-100\pi^2 k t/L^2} \sin \frac{10\pi x}{L}.$$

Illustration

$$u(x, t) = -2e^{-16\pi^2 k t/L^2} \sin \frac{4\pi x}{L} + 5e^{-100\pi^2 k t/L^2} \sin \frac{10\pi x}{L}.$$



Homework

- ▶ Read Section 1.6
- ▶ Exercises: 14, 18, 20