Separation of Variables

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn the approach of a fundamental technique for solving many PDEs, namely **separation of variables**.

This technique reduces the problem of finding the unknown dependent variable of the PDE u, which depends on n independent variables, to the problem of solving n ordinary differential equations each depending on a single independent variable.

We will assume the dependent variable is a **product solution.**

$$u(x, y, t) = X(x)Y(y)T(t)$$

Example

Apply the method of separation of variables to the equation

$$x^2 u_{xx} - 2y\, u_y = 0$$

and find a corresponding set of ordinary differential equations.

Solution (1 of 2)

- Assume $u(x, y) = X(x)Y(y)$ then

$$\begin{aligned} u_{xx} &= X''(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

- Substitute into the PDE.

$$\begin{aligned} x^2 u_{xx} - 2y\, u_y &= 0 \\ x^2 X''(x)Y(y) - 2yX(x)Y'(y) &= 0 \end{aligned}$$

- Divide both sides by $u(x, y) = X(x)Y(y)$.

$$\begin{aligned} \frac{x^2 X''(x)Y(y)}{X(x)Y(y)} - \frac{2yX(x)Y'(y)}{X(x)Y(y)} &= 0 \\ x^2 \frac{X''(x)}{X(x)} &= 2y\, \frac{Y'(y)}{Y(y)} \end{aligned}$$

Solution (2 of 2)

$$x^2 \frac{X''(x)}{X(x)} = 2y \frac{Y'(y)}{Y(y)}$$

Key observation: the left-hand side is a function of x while the right-hand side is a function of y. Since they are equal they must be constant.

$$x^2 \frac{X''(x)}{X(x)} = c = 2y \frac{Y'(y)}{Y(y)}$$

This implies

$$\begin{aligned} x^2 \tfrac{X''(x)}{X(x)} &= c \\ 2y \tfrac{Y'(y)}{Y(y)} &= c \end{aligned} \iff \begin{aligned} x^2 X''(x) - c\,X(x) &= 0 \\ 2y\,Y'(y) - c\,Y(y) &= 0 \end{aligned}$$

Example

Apply the method of separation of variables to the following equation and determine the corresponding set of ordinary differential equations.

$$u_{xx} + u_x + 2u_y - u \sin x = 0$$

Example

Determine if the method of separation of variables can be applied to the following partial differential equation. If so, determine the resulting ordinary differential equations.

$$u_x + (x+y)u_y = 0.$$

Example

Determine if the method of separation of variables can be applied to the partial differential equation below. If so, determine the ordinary differential equations in each variable which result.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad (\text{for } r > 0)$$

The dependent variable is bounded as $r \to 0$ and is 2π-periodic in θ.

Example: The Heat Equation

Consider the one-dimensional, homogeneous heat equation with Dirichlet boundary conditions and an initial condition as below.

$$\begin{aligned} u_t &= k\, u_{xx}, \quad 0 < x < L, \quad t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(L,t) &= 0, \quad t > 0 \\ u(x,0) &= f(x), \quad 0 \leq x \leq L \end{aligned}$$

Apply the method of separation of variables to this initial boundary value problem and determine the product solutions which satisfy the homogeneous boundary conditions.

Solution (1 of 8)

- Assume the product solution $u(x,t) = X(x)T(t)$.
- Differentiating and substituting into the heat equation yields

$$\begin{aligned} u_t &= k\, u_{xx} \\ X(x)T'(t) &= k\, X''(x)T(t) \\ \frac{1}{k}\frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = -c \end{aligned}$$

  where c is a constant and the minus sign is introduced for convenience.
- The resulting ODEs for x and t are

$$\begin{aligned} X''(x) + c\, X(x) &= 0 \\ T'(t) + c\, k\, T(t) &= 0. \end{aligned}$$

Solution (2 of 8)

Consider the ODE for x and the boundary conditions.

$$\begin{aligned} X''(x) + c\, X(x) &= 0 \\ u(0,t) = X(0)T(t) &= 0 \iff X(0) = 0 \\ u(L,t) = X(L)T(t) &= 0 \iff X(L) = 0 \end{aligned}$$

Find solutions to the ODE which satisfy the boundary conditions. Consider the three cases:

- $c = 0$,
- $c < 0$,
- $c > 0$.

Solution (3 of 8)

Case: $c = 0$.

$$\begin{aligned} X''(x) + c\, X(x) = X''(x) &= 0 \\ X(x) &= Ax + B \end{aligned}$$

When $x = 0$ we have $0 = X(0) = B$.

When $x = L$ we have $0 = X(L) = A\, L$ which implies $A = 0$. Thus when $c = 0$ we have only the trivial solution $X(x) = 0$.

Solution (4 of 8)

Case: $c < 0$. Let $c = -\lambda^2$ where $\lambda > 0$.

$$\begin{aligned} X''(x) + c\,X(x) = X''(x) - \lambda^2 X(x) &= 0 \\ X(x) &= A\cosh(\lambda x) + B\sinh(\lambda x) \end{aligned}$$

When $x = 0$ we have $0 = X(0) = A$.

When $x = L$ we have $0 = X(L) = B\sinh(\lambda L)$ which implies $B = 0$. Thus when $c < 0$ we have only the trivial solution $X(x) = 0$.

Solution (5 of 8)

Case: $c > 0$. Let $c = \lambda^2$ where $\lambda > 0$.

$$\begin{aligned} X''(x) + c\,X(x) = X''(x) + \lambda^2 X(x) &= 0 \\ X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \end{aligned}$$

When $x = 0$ we have $0 = X(0) = A$.

When $x = L$ we have $0 = X(L) = B\sin(\lambda L)$ which implies

$$\begin{aligned} \lambda L &= n\pi \\ \lambda \equiv \lambda_n &= \frac{n\pi}{L} \end{aligned}$$

for $n \in \mathbb{N}$. Thus when $c = n^2\pi^2/L^2$ for $n \in \mathbb{N}$ we have the nontrivial solutions

$$X_n(x) = \sin\frac{n\pi x}{L}.$$

Function $X_n(x)$ is called an **eigenfunction** corresponding to **eigenvalue** $n^2\pi^2/L^2$.

Solution (6 of 8)

Using the known eigenvalue in the ODE for t yields

$$\begin{aligned} T'(t) + \frac{n^2\pi^2 k}{L^2} T(t) &= 0 \\ T(t) \equiv T_n(t) &= C_n e^{-n^2\pi^2 k\,t/L^2} \end{aligned}$$

for $n \in \mathbb{N}$.

The product solutions which satisfy the boundary conditions have the form

$$u_n(x,t) = X_n(x)T_n(t) = B_n e^{-n^2\pi^2 k\,t/L^2} \sin\left(\frac{n\pi x}{L}\right).$$

These are called **fundamental solutions**.

Solution (7 of 8)

Using the Principle of Superposition, a finite linear combination of fundamental solutions will likewise solve the PDE and satisfy the BCs.

$$u(x,t) = \sum_{n=1}^{N} B_n e^{-n^2\pi^2 k\,t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

Now consider the initial condition:

$$\begin{aligned} u(x,0) &= f(x) \\ \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi x}{L}\right) &= f(x) \end{aligned}$$

As long as $f(x)$ contains a finite sum of sine functions of the appropriate periods we can equate coefficients and solve for the B_n's.

Solution (8 of 8)

Take the case in which

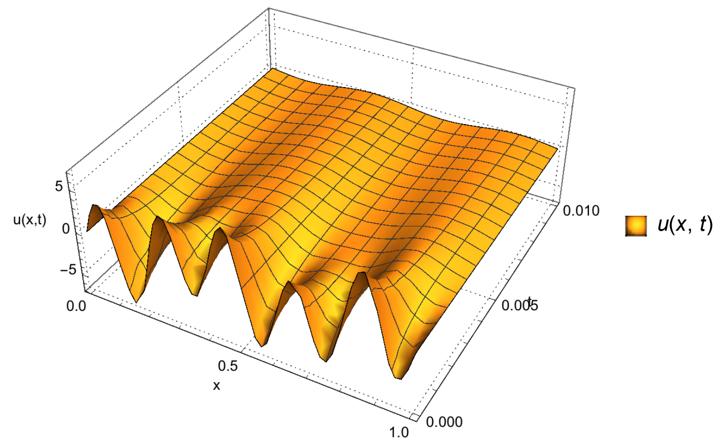
$$f(x) = -2\sin\frac{4\pi x}{L} + 5\sin\frac{10\pi x}{L}.$$

Then $B_4 = -2$, $B_{10} = 5$ and all other coefficients are 0.

$$u(x,t) = -2e^{-16\pi^2 k\,t/L^2}\sin\frac{4\pi x}{L} + 5e^{-100\pi^2 k\,t/L^2}\sin\frac{10\pi x}{L}.$$

Illustration

$$u(x,t) = -2e^{-16\pi^2 k\,t/L^2}\sin\frac{4\pi x}{L} + 5e^{-100\pi^2 k\,t/L^2}\sin\frac{10\pi x}{L}.$$



Homework

- Read Section 1.6
- Exercises: 14, 18, 20