# Separation of Variables

*Partial Differential Equations*

J Robert Buchanan

Department of Mathematics

Fall 2025

# Objectives

In this lesson we will learn the approach of a fundamental technique for solving many PDEs, namely **separation of variables.**

This technique reduces the problem of finding the unknown dependent variable of the PDE $u$, which depends on $n$ independent variables, to the problem of solving $n$ ordinary differential equations each depending on a single independent variable.

We will assume the dependent variable is a **product solution.**

$$u(x, y, t) = X(x)Y(y)T(t)$$

# Example

Apply the method of separation of variables to the equation

$$x^2 u_{xx} - 2y\, u_y = 0$$

and find a corresponding set of ordinary differential equations.

# Solution (1 of 2)

- Assume $u(x, y) = X(x)Y(y)$ then

$$\begin{aligned} u_{xx} &= X''(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

# Solution (1 of 2)

- Assume $u(x, y) = X(x)Y(y)$ then

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- Substitute into the PDE.

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$$\begin{aligned} x^2 u_{xx} - 2y\, u_y &= 0 \\ x^2 X''(x)Y(y) - 2yX(x)Y'(y) &= 0 \end{aligned}$$

- Divide both sides by $u(x, y) = X(x)Y(y)$.

$$\frac{x^2 X''(x)Y(y)}{X(x)Y(y)} - \frac{2yX(x)Y'(y)}{X(x)Y(y)} = 0$$

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- Divide both sides by $u(x,y) = X(x)Y(y)$.

$$\begin{aligned} \frac{x^2 X''(x)Y(y)}{X(x)Y(y)} - \frac{2yX(x)Y'(y)}{X(x)Y(y)} &= 0 \\ x^2 \frac{X''(x)}{X(x)} &= 2y\, \frac{Y'(y)}{Y(y)} \end{aligned}$$

# Solution (2 of 2)

$$x^2 \frac{X''(x)}{X(x)} = 2y \frac{Y'(y)}{Y(y)}$$

**Key observation**: the left-hand side is a function of $x$ while the right-hand side is a function of $y$. Since they are equal they must be constant.

# Solution (2 of 2)

$$x^2 \frac{X''(x)}{X(x)} = 2y\,\frac{Y'(y)}{Y(y)}$$

**Key observation**: the left-hand side is a function of $x$ while the right-hand side is a function of $y$. Since they are equal they must be constant.

$$x^2 \frac{X''(x)}{X(x)} = c = 2y\,\frac{Y'(y)}{Y(y)}$$

This implies

$$\begin{aligned} x^2 \tfrac{X''(x)}{X(x)} &= c \\ 2y\,\tfrac{Y'(y)}{Y(y)} &= c \end{aligned}$$

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This implies

$$\begin{array}{rcl} x^2 \frac{X''(x)}{X(x)} & = & c \\ 2y \, \frac{Y'(y)}{Y(y)} & = & c \end{array} \iff \begin{array}{rcl} x^2 X''(x) - c \, X(x) & = & 0 \\ 2y \, Y'(y) - c \, Y(y) & = & 0 \end{array}$$

# Example

Apply the method of separation of variables to the following equation and determine the corresponding set of ordinary differential equations.

$$u_{xx} + u_x + 2u_y - u \sin x = 0$$

# Solution (1 of 2)

- Assume $u(x,y) = X(x)Y(y)$ then

$$\begin{aligned} u_x &= X'(x)Y(y) \\ u_{xx} &= X''(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

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- Assume $u(x, y) = X(x)Y(y)$ then

$$\begin{aligned} u_x &= X'(x)Y(y) \\ u_{xx} &= X''(x)Y(y) \\ u_y &= X(x)Y'(y). \end{aligned}$$

- Substitute into the PDE.

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- Substitute into the PDE.

$$\begin{aligned} u_{xx} + u_x + 2u_y - u \sin x &= 0 \\ X''(x)Y(y) + X'(x)Y(y) + 2X(x)Y'(y) - X(x)Y(y) \sin x &= 0 \end{aligned}$$

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- Divide both sides by $u(x, y) = X(x)Y(y)$.

$$\frac{X''(x)Y(y)}{X(x)Y(y)} + \frac{X'(x)Y(y)}{X(x)Y(y)} + \frac{2X(x)Y'(y)}{X(x)Y(y)} - \frac{X(x)Y(y) \sin x}{X(x)Y(y)} = 0$$

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- Substitute into the PDE.

$$\begin{aligned} u_{xx} + u_x + 2u_y - u \sin x &= 0 \\ X''(x)Y(y) + X'(x)Y(y) + 2X(x)Y'(y) - X(x)Y(y)\sin x &= 0 \end{aligned}$$

- Divide both sides by $u(x, y) = X(x)Y(y)$.

$$\begin{aligned} \frac{X''(x)Y(y)}{X(x)Y(y)} + \frac{X'(x)Y(y)}{X(x)Y(y)} + \frac{2X(x)Y'(y)}{X(x)Y(y)} - \frac{X(x)Y(y)\sin x}{X(x)Y(y)} &= 0 \\ \frac{X''(x)}{X(x)} + \frac{X'(x)}{X(x)} - \sin x &= -2\,\frac{Y'(y)}{Y(y)} \end{aligned}$$

# Solution (2 of 2)

$$\frac{X''(x)}{X(x)} + \frac{X'(x)}{X(x)} - \sin x = -2\,\frac{Y'(y)}{Y(y)}$$

Since the left-hand side is a function of $x$ while the right-hand side is a function of $y$. Since they are equal they must be constant.

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This implies

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This implies

$$\begin{array}{rcl} \frac{X''(x)}{X(x)} + \frac{X'(x)}{X(x)} - \sin x & = & c \\ -2\,\frac{Y'(y)}{Y(y)} & = & c \end{array} \iff \begin{array}{rcl} X''(x) + X'(x) - X(x)\sin x & = & c\,X(x) \\ 2\,Y'(y) + c\,Y(y) & = & 0 \end{array}$$

# Example

Determine if the method of separation of variables can be applied to the following partial differential equation. If so, determine the resulting ordinary differential equations.

$$u_x + (x + y)u_y = 0.$$

# Solution

- Assume $u(x, y) = X(x)Y(y)$ then

$$
\begin{aligned}
u_x &= X'(x)Y(y) \\
u_y &= X(x)Y'(y).
\end{aligned}
$$

# Solution

- Assume $u(x, y) = X(x)Y(y)$ then

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- Substitute into the PDE.

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$$\begin{aligned} u_x + (x+y)u_y &= 0 \\ X'(x)Y(y) + (x+y)X(x)Y'(y) &= 0 \end{aligned}$$

- Divide both sides by $u(x, y) = X(x)Y(y)$.

$$\begin{aligned} \frac{X'(x)Y(y)}{X(x)Y(y)} + \frac{(x+y)X(x)Y'(y)}{X(x)Y(y)} &= 0 \\ \frac{X'(x)}{X(x)} + \frac{(x+y)Y'(y)}{Y(y)} &= 0 \end{aligned}$$

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$$\begin{aligned} \frac{X'(x)Y(y)}{X(x)Y(y)} + \frac{(x+y)X(x)Y'(y)}{X(x)Y(y)} &= 0 \\ \frac{X'(x)}{X(x)} + \frac{(x+y)Y'(y)}{Y(y)} &= 0 \end{aligned}$$

It is not possible to separate the variables in this example.

# Example

Determine if the method of separation of variables can be applied to the partial differential equation below. If so, determine the ordinary differential equations in each variable which result.

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{(for } r > 0\text{)}$$

The dependent variable is bounded as $r \to 0$ and is $2\pi$-periodic in $\theta$.

# Solution (1 of 4)

- Let $u(r,\theta) = R(r)T(\theta)$, then

$$\begin{aligned} u_r &= R'(r)T(\theta) \\ u_{rr} &= R''(r)T(\theta) \\ u_{\theta\theta} &= R(r)T''(\theta). \end{aligned}$$

- Substitute into the PDE.

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0 \\ R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) &= 0 \end{aligned}$$

- Multiply both sides of the equation by $r^2/(R(r)T(\theta))$.

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{T''(\theta)}{T(\theta)} = 0$$

- Separate the variables.

# Solution (2 of 4)

$$r^2 \frac{R''(r)}{R(r)} + r\,\frac{R'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)}$$

# Solution (2 of 4)

$$r^2 \frac{R''(r)}{R(r)} + r\,\frac{R'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = c$$

where $c$ is a constant.

The ordinary differential equations for $r$ and $\theta$ are thus

$$\begin{aligned} r^2 R''(r) + r\,R'(r) - c\,R(r) &= 0 \\ T''(\theta) + c\,T(\theta) &= 0. \end{aligned}$$

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The ordinary differential equations for $r$ and $\theta$ are thus

$$\begin{aligned} r^2 R''(r) + r\,R'(r) - c\,R(r) &= 0 \\ T''(\theta) + c\,T(\theta) &= 0. \end{aligned}$$

The equation in $\theta$ is a second-order linear, constant coefficient, linear, homogeneous ODE.

# Solution (3 of 4)

If $c \geq 0$ then the ODE

$$T''(\theta) + c\,T(\theta) = 0$$

has solutions

$$T(\theta) = A\cos(\sqrt{c}\,\theta) + B\sin(\sqrt{c}\,\theta).$$

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$$T(\theta) = A\cos(\sqrt{c}\,\theta) + B\sin(\sqrt{c}\,\theta).$$

The solution $u(r, \theta)$ should be $2\pi$-periodic in $\theta$, thus

$$\frac{2\pi}{\sqrt{c}} = \frac{2\pi}{n} \iff c = n^2$$

where $n \in \mathbb{N}$ or $c = 0$.

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where $n \in \mathbb{N}$ or $c = 0$.

Thus the solutions for $T(\theta)$ can be summarized as

$$T_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$$

for $n = 0, 1, 2, \ldots$.

# Solution (4 of 4)

Since $c = n^2$ for $n = 0, 1, 2, \dots$ then the ODE for $r$ can be written as

$$r^2 R''(r) + r\,R'(r) - n^2 R(r) = 0$$

which is Euler's equation. The solutions are

$$\begin{aligned} R_0(r) &= C_0 + D_0 \ln r \quad (\text{for } n = 0) \\ R_n(r) &= C_n r^n + D_n r^{-n} \quad (\text{for } n = 1, 2, \dots) \end{aligned}$$

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The solutions are supposed to be bounded as $r \to 0$ and thus $D_0 = D_n = 0$ for $n = 1, 2, \ldots$. Consequently

$$u_n(r, \theta) = r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

for $n = 0, 1, 2, \ldots$.

# Example: The Heat Equation

Consider the one-dimensional, homogeneous heat equation with Dirichlet boundary conditions and an initial condition as below.

$$\begin{aligned} u_t &= k\,u_{xx}, \quad 0 < x < L, \quad t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(L,t) &= 0, \quad t > 0 \\ u(x,0) &= f(x), \quad 0 \le x \le L \end{aligned}$$

Apply the method of separation of variables to this initial boundary value problem and determine the product solutions which satisfy the homogeneous boundary conditions.

# Solution (1 of 8)

- Assume the product solution $u(x,t) = X(x)T(t)$.
- Differentiating and substituting into the heat equation yields

$$\begin{aligned} u_t &= k\, u_{xx} \\ X(x)T'(t) &= k\, X''(x)T(t) \\ \frac{1}{k}\,\frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = -c \end{aligned}$$

  where $c$ is a constant and the minus sign is introduced for convenience.
- The resulting ODEs for $x$ and $t$ are

$$\begin{aligned} X''(x) + c\,X(x) &= 0 \\ T'(t) + c\,k\,T(t) &= 0. \end{aligned}$$

# Solution (2 of 8)

Consider the ODE for $x$ and the boundary conditions.

$$
\begin{aligned}
X''(x) + c\,X(x) &= 0 \\
u(0,t) = X(0)T(t) &= 0 \iff X(0) = 0 \\
u(L,t) = X(L)T(t) &= 0 \iff X(L) = 0
\end{aligned}
$$

Find solutions to the ODE which satisfy the boundary conditions.
Consider the three cases:

- $c = 0$,
- $c < 0$,
- $c > 0$.

# Solution (3 of 8)

**Case**: $c = 0$.

$$\begin{aligned} X''(x) + c\,X(x) = X''(x) &= 0 \\ X(x) &= Ax + B \end{aligned}$$

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**Case**: $c = 0$.

$$\begin{aligned} X''(x) + c\,X(x) = X''(x) &= 0 \\ X(x) &= Ax + B \end{aligned}$$

When $x = 0$ we have $0 = X(0) = B$.

When $x = L$ we have $0 = X(L) = A\,L$ which implies $A = 0$. Thus when $c = 0$ we have only the trivial solution $X(x) = 0$.

# Solution (4 of 8)

**Case**: $c < 0$. Let $c = -\lambda^2$ where $\lambda > 0$.

$$\begin{aligned} X''(x) + c\,X(x) = X''(x) - \lambda^2 X(x) &= 0 \\ X(x) &= A\cosh(\lambda x) + B\sinh(\lambda x) \end{aligned}$$

# Solution (4 of 8)

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When $x = 0$ we have $0 = X(0) = A$.

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When $x = 0$ we have $0 = X(0) = A$.

When $x = L$ we have $0 = X(L) = B\sinh(\lambda L)$ which implies $B = 0$.
Thus when $c < 0$ we have only the trivial solution $X(x) = 0$.

# Solution (5 of 8)

**Case**: $c > 0$. Let $c = \lambda^2$ where $\lambda > 0$.

$$\begin{aligned} X''(x) + c\,X(x) = X''(x) + \lambda^2 X(x) &= 0 \\ X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \end{aligned}$$

# Solution (5 of 8)

**Case**: $c > 0$. Let $c = \lambda^2$ where $\lambda > 0$.

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# Solution (5 of 8)

**Case**: $c > 0$. Let $c = \lambda^2$ where $\lambda > 0$.

$$X''(x) + c\,X(x) = X''(x) + \lambda^2 X(x) = 0$$
$$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

When $x = 0$ we have $0 = X(0) = A$.

When $x = L$ we have $0 = X(L) = B\sin(\lambda L)$ which implies

$$\lambda L = n\pi$$
$$\lambda \equiv \lambda_n = \frac{n\pi}{L}$$

for $n \in \mathbb{N}$. Thus when $c = n^2\pi^2/L^2$ for $n \in \mathbb{N}$ we have the nontrivial solutions

$$X_n(x) = \sin\frac{n\pi x}{L}.$$

Function $X_n(x)$ is called an **eigenfunction** corresponding to **eigenvalue** $n^2\pi^2/L^2$.

# Solution (6 of 8)

Using the known eigenvalue in the ODE for $t$ yields

$$T'(t) + \frac{n^2\pi^2 k}{L^2} T(t) = 0$$
$$T(t) \equiv T_n(t) = C_n e^{-n^2\pi^2 k\, t/L^2}$$

for $n \in \mathbb{N}$.

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for $n \in \mathbb{N}$.

The product solutions which satisfy the boundary conditions have the form

$$u_n(x,t) = X_n(x) T_n(t) = B_n e^{-n^2\pi^2 k\, t/L^2} \sin\left(\frac{n\pi x}{L}\right).$$

These are called **fundamental solutions**.

# Solution (7 of 8)

Using the Principle of Superposition, a finite linear combination of fundamental solutions will likewise solve the PDE and satisfy the BCs.

$$u(x,t) = \sum_{n=1}^{N} B_n e^{-n^2\pi^2 k\,t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

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Now consider the initial condition:

$$\begin{aligned} u(x,0) &= f(x) \\ \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi x}{L}\right) &= f(x) \end{aligned}$$

As long as $f(x)$ contains a finite sum of sine functions of the appropriate periods we can equate coefficients and solve for the $B_n$'s.

# Solution (8 of 8)

Take the case in which

$$f(x) = -2 \sin \frac{4\pi x}{L} + 5 \sin \frac{10\pi x}{L}.$$

Then $B_4 = -2$, $B_{10} = 5$ and all other coefficients are 0.

$$u(x,t) = -2e^{-16\pi^2 k t/L^2} \sin \frac{4\pi x}{L} + 5e^{-100\pi^2 k t/L^2} \sin \frac{10\pi x}{L}.$$

# Illustration

$$u(x,t) = -2e^{-16\pi^2 k\, t/L^2} \sin \frac{4\pi\, x}{L} + 5e^{-100\pi^2 k\, t/L^2} \sin \frac{10\pi\, x}{L}.$$