# First-Order Linear PDEs

MATH 467 *Partial Differential Equations*

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## Objectives

In this lesson we will learn:

- to classify first-order partial differential equations as either linear or quasilinear,
- to solve linear first-order partial differential equations,

## Transport Equations

The general form of a first-order scalar PDE is

$$u_x + \nabla \cdot f(x, y, u, \nabla u) = g(x, y, u)$$

where

- $\nabla \cdot f$ denotes the divergence of $f$, and
- $\nabla u$ denotes the gradient of $u$.

## Classification

A first-order PDE is **linear** if it can be written as

$$a(x, y)u_x(x, y) + b(x, y)u_y(x, y) = c(x, y)u(x, y) + d(x, y).$$

A first-order PDE is **semilinear** if it can be written as

$$a(x, y)u_x(x, y) + b(x, y)u_y(x, y) = c(x, y, u).$$

A first-order PDE is **quasilinear** if it can be written as

$$a(x, y, u)u_x(x, y) + b(x, y, u)u_y(x, y) = c(x, y, u).$$

## Simple Case

Let $c$ be a constant and consider

$$u_x + c\,u_y = 0$$
$$\langle u_x, u_y \rangle \cdot \langle 1, c \rangle = 0$$
$$D_{\langle 1,c \rangle} u(x,t) = 0.$$

**Remarks**:

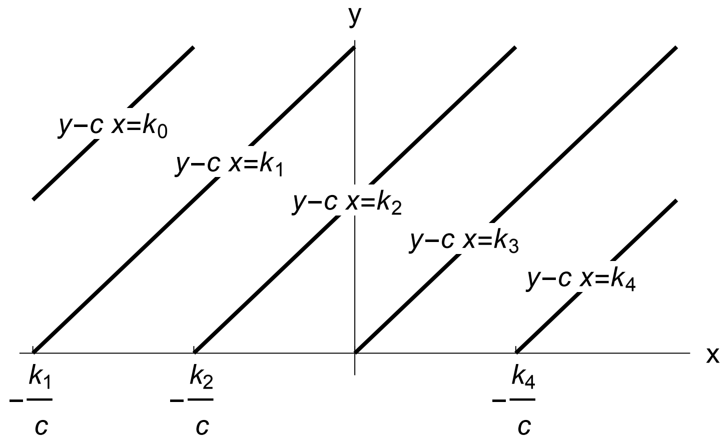
- This can be interpreted as stating the directional derivative of $u(x,y)$ in the direction of vector $\langle 1, c\rangle$ is 0.
- Function $u(x,y)$ is constant along lines of the form $y - c\,x = k$.
- Function $u(x,y) \equiv f(k)$ where $f$ is an arbitrary differentiable function.

## Illustration

The solutions of

$$u_x + c\,u_y = 0$$

must be constant along lines of the form $y - c\,x = k$.




## Confirmation of Solution

Let $u(x,y) = f(y - c\,x)$ then

$$u_x = -c\,f'(y - c\,x)$$
$$u_y = f'(y - c\,x)$$

and

$$u_x + c\,u_y = -c\,f'(y - c\,x) + c\,f'(y - c\,x) = 0.$$

Functions of the form $u(x,y) = f(y - c\,x)$ are referred to as the **general solutions** to this simple PDE.

## Initial Conditions

If the value of $u(x,y)$ along the line where $x = 0$ is specified, then an initial condition of the form $u(0,y) = \phi(y)$ further specifies the solution.

$$u(x,y) = f(y - c\,x) \text{ (general solution)}$$
$$u(0,y) = f(y) = \phi(y) \text{ (initial condition)}$$
$$u(x,y) = \phi(y - c\,x) \text{ (solution to IVP)}$$

# General First-Order Linear PDE

The solution technique used previously can be extended to the **general first-order linear PDE** of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y),$$

for $(x,y) \in D \subset \mathbb{R}^2$.

# Vector Field (1 of 3)

Suppose $(x,y) \equiv (x(t), y(t))$ where $t$ is a parameter, then

$$\frac{d}{dt}[u(x,y)] = u_x x'(t) + u_y y'(t).$$

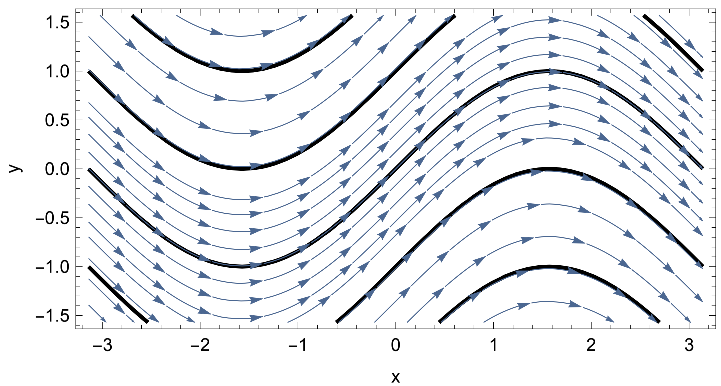
Consider

$$\begin{aligned} a(x,y)u_x + b(x,y)u_y &= c(x,y)u + d(x,y) \text{ as} \\ \frac{dx}{dt}u_x + \frac{dy}{dt}u_y &= c(x,y)u + d(x,y) \text{ where} \\ \frac{dx}{dt} &= a(x,y) \\ \frac{dy}{dt} &= b(x,y). \end{aligned}$$

**Remark**: the parametric curve $(x(t), y(t))$ is an **integral curve** of the **vector field** $\langle a(x,y), b(x,y)\rangle$ for all $(x,y) \in D$.

# Vector Field (2 of 3)

A vector field and integral curves:




# Vector Field (3 of 3)

If $\dfrac{dx}{dt} = a(x,y)$ and $\dfrac{dy}{dt} = b(x,y)$ then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b(x,y)}{a(x,y)}.$$

Suppose the implicit form of the solution of this ODE is $\phi(x,y) = k$ where $k$ is an arbitrary constant.

Along the curves defined by $\phi(x,y) = k$ the general first-order linear PDE

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y)$$

becomes

$$\begin{aligned} u_x + u_y\frac{b(x,y)}{a(x,y)} &= \frac{c(x,y)u + d(x,y)}{a(x,y)} \\ u_x + u_y\frac{dy}{dx} &= \frac{c(x,y)u + d(x,y)}{a(x,y)} \\ \frac{du}{dx} &= \frac{c(x,y)u + d(x,y)}{a(x,y)}. \end{aligned}$$

## Ordinary Differential Equation

$$\frac{du}{dx} = \frac{c(x,y)u + d(x,y)}{a(x,y)}$$

is an ordinary differential equation for $u$ as a function of variable $x$ (since $y \equiv y(x)$) and arbitrary constant $k$.

Solving this ODE yields $u \equiv u(x,k) \equiv u(x,y)$ since $k = \phi(x,y)$.

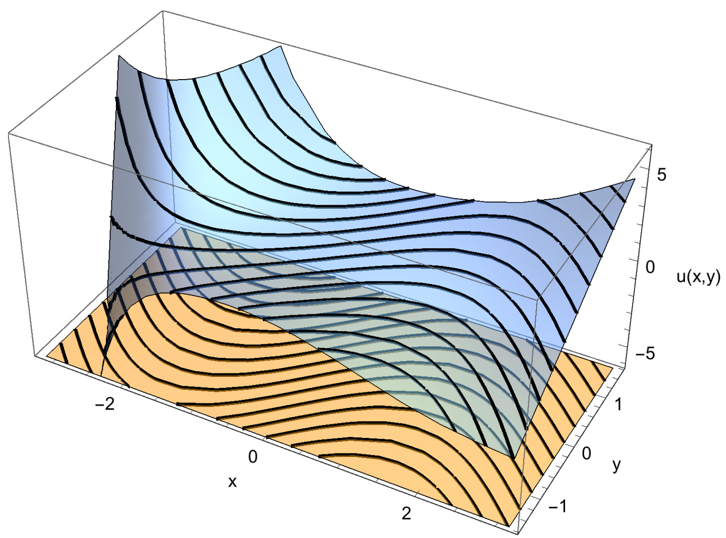
## Characteristics

- The ordinary differential equations:

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$
$$\frac{du}{dt} = c(x,y)u + d(x,y)$$

  are called the **characteristic equations**.
- The solution curves $(x(t), y(t), u(x(t), y(t)))$ to the characteristic equations are called the **characteristic curves**.
- The curves in the $xy$-plane of the form $(x(t), y(t))$ are called **characteristics**.

## Characteristic Curves *vs.* Characteristics




Characteristics are curves in the $xy$-plane and characteristic curves are curves on the surface $u(x,y)$.

## Example

Consider the initial value problem:

$$2u_x + 3u_y - 4u = 0$$
$$u(x,0) = \sin x.$$

1. Find the general solution to the PDE using the method of characteristics.
2. Find the solution to the IVP.

## Solution (1 of 2)

$$2u_x + 3u_y - 4u = 0$$

In this PDE we can let $a(x, y) = 2$ and $b(x, y) = 3$, thus

$$\frac{dy}{dx} = \frac{3}{2} \implies y = \frac{3}{2}x + \hat{k} \iff 2y - 3x = k.$$

**Comment**: the characteristics for this PDE are straight lines of the form $2y - 3x = k$.

Think of $u(x, y) = u(x, y(x))$ so that

$$\frac{du}{dx} = u_x + u_y \frac{dy}{dx} = u_x + \frac{3}{2}u_y = 2u$$

which implies

$$u(x, y) = f(k)e^{2x} = f(2y - 3x)e^{2x}$$

where $f$ is an arbitrary differentiable function.

## Solution (2 of 2)

In order to satisfy the initial condition:

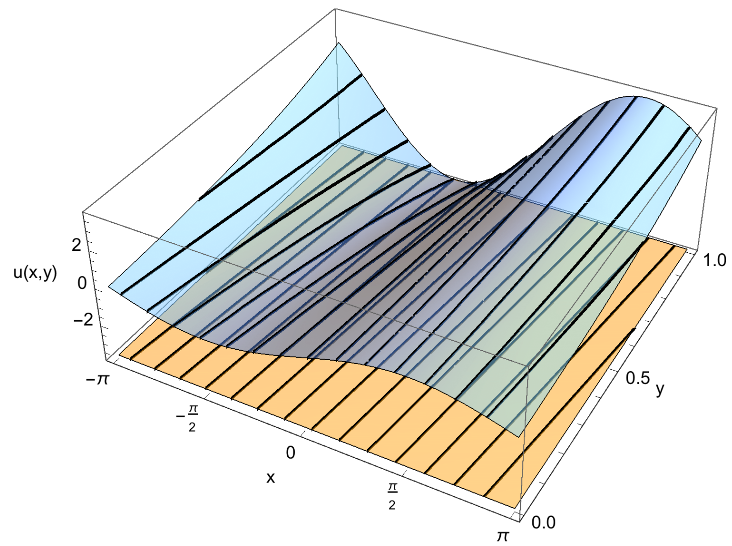
$$\begin{aligned} u(x, 0) &= \sin x \\ f(-3x)e^{2x} &= \sin x \\ f(-3x) &= e^{-2x} \sin x \\ f(z) &= -e^{2z/3} \sin(z/3). \end{aligned}$$

Thus the solution to the IVP is

$$u(x, y) = -e^{2(2y-3x)/3} \sin((2y - 3x)/3)e^{2x} = -e^{4y/3} \sin\left(\frac{2}{3}y - x\right).$$

## Illustration

$$u(x, y) = -e^{4y/3} \sin\left(\frac{2}{3}y - x\right)$$




## Example

Find the general solution to the first-order linear PDE:

$$-y\, u_x + x\, u_y = 0.$$

## Solution (1 of 2)

If we let $a(x,y) = -y$ and $b(x,y) = x$ then the characteristics satisfy the ODE

$$
\begin{aligned}
\frac{dy}{dx} &= -\frac{x}{y} \\
y\,dy &= -x\,dx \\
\int y\,dy &= -\int x\,dx \\
\frac{1}{2}y^2 &= -\frac{1}{2}x^2 + \hat{k} \\
x^2 + y^2 &= k.
\end{aligned}
$$

In this example the characteristics are circles of radius $\sqrt{k}$ for $k \geq 0$.

## Solution (2 of 2)

Assuming $u(x,y) = u(x, y(x))$ then

$$
\begin{aligned}
\frac{du}{dx} &= u_x + u_y \frac{dy}{dx} \\
&= u_x - \frac{x}{y} u_y \\
\frac{du}{dx} &= 0 \\
u &= f(k) \\
u(x,y) &= f(x^2 + y^2)
\end{aligned}
$$

where $f$ is an arbitrary differentiable function.

## Example

Solve the initial value problem:

$$
\begin{aligned}
2x\,y\,u_x + u_y - u &= 0 \\
u(x,0) &= x,
\end{aligned}
$$

for $x > 0$ and $y > 0$.

## Solution (1 of 2)

Let $a(x,y) = 2x\,y$ and $b(x,y) = 1$ then

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2x\,y} \text{ (separable ODE)} \\
y\,dy &= \frac{1}{2x}\,dx \\
\int y\,dy &= \int \frac{1}{2x}\,dx \\
\frac{1}{2}y^2 &= \frac{1}{2}\ln x + \hat{k} \\
y^2 - \ln x &= k.
\end{aligned}
$$

Thus curves of the form $y = \sqrt{k + \ln x}$ are the characteristics of the solution.

# Solution (2 of 2)

To find the general solution to the PDE:

$$\begin{aligned}
2xy\,u_x + u_y - u &= 0 \\
u_x + \frac{1}{2xy} u_y &= \frac{u}{2xy} \\
\frac{du}{dx} &= \frac{u}{2x\sqrt{k+\ln x}} \text{ (separable ODE)} \\
\frac{1}{u}\,du &= \frac{1}{2x\sqrt{k+\ln x}}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{2x\sqrt{k+\ln x}}\,dx \text{ (substitute } v = k+\ln x\text{)} \\
\ln u &= \sqrt{k+\ln x} + \ln f(k) \\
u(x,y) &= f(k)e^{\sqrt{k+\ln x}} = f(y^2 - \ln x)e^{y}
\end{aligned}$$

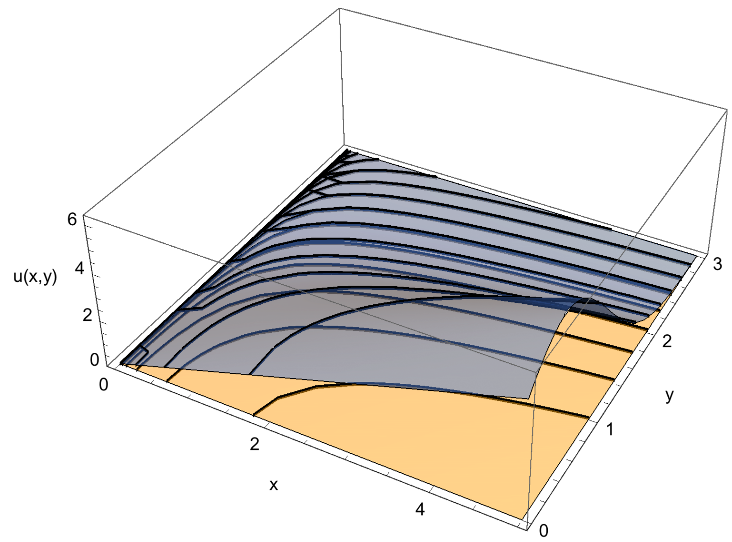
where $f$ is an arbitrary differentiable function.

Since $u(x,0) = x$ then

$$x = f(-\ln x) \iff f(x) = e^{-x} \implies u(x,y) = e^{-y^2+\ln x}e^{y} = x\,e^{y-y^2}.$$

# Illustration

$$u(x,y) = x\,e^{y-y^2}$$




# Example

Show that the problem

$$\begin{aligned}
-y\,u_x + xu_y &= 0 \\
u(x,0) &= 3x
\end{aligned}$$

has no solution.

# Justification

From a previous example we know the characteristics are the curves of the form $x^2 + y^2 = k$ and the general solution has the form

$$u(x,y) = f(x^2 + y^2)$$

where $f$ is an arbitrary differentiable function.

If $x \neq 0$ then

$$u(-x,0) = -3x \neq 3x = u(x,0)$$

but

$$u(-x,0) = f((-x)^2) = f(x^2) = u(x,0)$$

which is a contradiction.

## Characteristics

The side condition is specified on the $x$-axis which intersects each characteristic curve twice (at $x = -\sqrt{k}$ and $x = \sqrt{k}$). The side condition specifies two different values of the solution, but the solution must be constant on each characteristic curve.

## Example

Consider the first-order, linear PDE

$$u_x + (\cos x)u_y + u = x\,y.$$

Find the general solution to this PDE.

## Solution (1 of 4)

Let $a(x, y) = 1$ and $b(x, y) = \cos x$, then the characteristics satisfy the ODE

$$\begin{aligned} \frac{dy}{dx} &= \cos x \\ dy &= \cos x\,dx \\ \int 1\,dy &= \int \cos x\,dx \\ y &= \sin x + k. \end{aligned}$$

In this example the characteristics are curves of the form $y - \sin x = k$.

## Solution (2 of 4)

Assuming $u(x, y) = u(x, y(x))$ then

$$\begin{aligned} u_x + (\cos x)u_y + u &= x\,y \\ \frac{du}{dx} &= x\,y - u = x(\sin x + k) - u \\ \frac{du}{dx} + u &= x \sin x + k\,x. \end{aligned}$$

This is a first-order, linear ODE which can be solved by multiplying both sides by the integrating factor $e^x$ and integrating.

## Solution (3 of 4)

$$\begin{aligned}
\frac{du}{dx} + u &= x \sin x + k\,x \\
\frac{d}{dx}\left[u\,e^x\right] &= x\,e^x \sin x + k\,x\,e^x \\
\int d\left[u\,e^x\right] &= \int (x\,e^x \sin x + k\,x\,e^x)\,dx \\
u\,e^x &= \frac{1}{2} e^x \left(2k(x-1) - (x-1)\cos x + x \sin x\right) + f(k) \\
u &= \frac{1}{2}\left(2k(x-1) - (x-1)\cos x + x \sin x\right) + e^{-x} f(k)
\end{aligned}$$

where $f$ is an arbitrary differentiable function.

## Solution (4 of 4)

$$\begin{aligned}
u &= \frac{1}{2}\left(2k(x-1) - (x-1)\cos x + x \sin x\right) + e^{-x} f(k) \\
u(x,y) &= (x-1)(y - \sin x) - \frac{x-1}{2}\cos x + \frac{x}{2}\sin x + e^{-x} f(y - \sin x)
\end{aligned}$$

## Example

Consider the first-order, linear PDE and side condition

$$\begin{aligned}
y\,u_x - 4x\,u_y &= 2x\,y \\
u(x,0) &= x^4
\end{aligned}$$

Find a particular solution to this PDE which satisfies the side condition.

## Solution (1 of 3)

Let $a(x,y) = y$ and $b(x,y) = -4x$, then the characteristics satisfy the ODE

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{4x}{y} \text{ (separable ODE)} \\
y\,dy &= -4x\,dx \\
\int y\,dy &= -\int 4x\,dx \\
\frac{1}{2}y^2 &= -2x^2 + \hat{k}.
\end{aligned}$$

In this example the characteristics are curves of the form $y^2 + 4x^2 = k$.

## Solution (2 of 3)

Assuming $u(x,y) = u(x, y(x))$ then

$$\begin{aligned}
y\,u_x - 4x\,u_y &= 2x\,y \\
u_x - \frac{4x}{y}\,u_y &= 2x \\
\frac{du}{dx} &= 2x \text{ (separable ODE)} \\
du &= 2x\,dx \\
\int 1\,du &= \int 2x\,dx \\
u &= x^2 + f(k) \\
u(x,y) &= x^2 + f(4x^2 + y^2)
\end{aligned}$$

where $f$ is an arbitrary differentiable function.

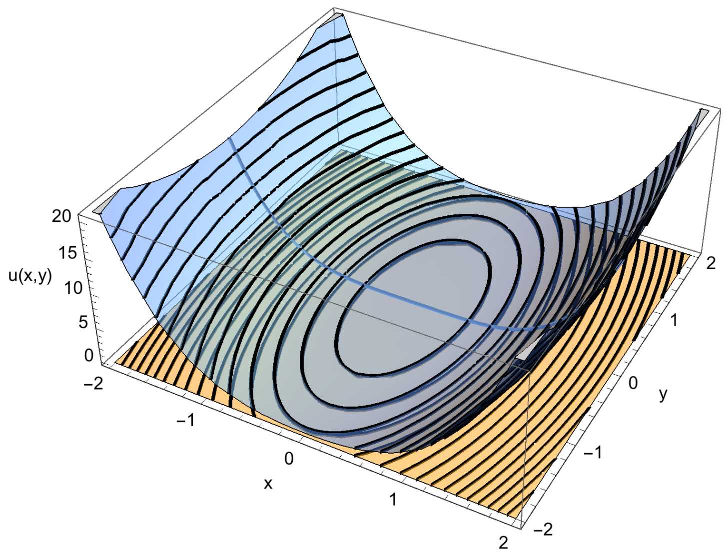
## Solution (3 of 3)

To satisfy the side condition we need

$$\begin{aligned}
u(x,0) &= x^2 + f(4x^2) \\
x^4 &= x^2 + f(4x^2) \\
f(4x^2) &= x^4 - x^2 \\
f(z) &= \frac{z^2}{16} - \frac{z}{4} \\
u(x,y) &= x^2 + \frac{(4x^2+y^2)^2}{16} - \frac{4x^2+y^2}{4}
\end{aligned}$$

## Illustration

$$u(x,y) = x^2 + \frac{(4x^2+y^2)^2}{16} - \frac{4x^2+y^2}{4}$$




## Example

Consider the first-order, linear PDE and side condition

$$\begin{aligned}
y\,u_x - 4x\,u_y &= 2x\,y \\
u(x,0) &= x^3
\end{aligned}$$

Show there is no solution to this PDE which satisfies the side condition.

## Justification

From the work done previously we know the general solution to the PDE takes the form

$$u(x, y) = x^2 + f(4x^2 + y^2)$$

where $f$ is an arbitrary differentiable function.

Suppose $u(x, y)$ is such that $u(x, 0) = x^3$, then for any $x$

$$u(-x, 0) = (-x)^2 + f(4(-x)^2 + 0^2) = x^2 + f(4x^2 + 0^2) = u(x, 0)$$

However, when $x \neq 0$,

$$u(-x, 0) = (-x)^3 \neq x^3 = u(x, 0)$$

which is a contradiction.

## Homework

- Read Section 2.1
- Exercises: 1–5