

# First-Order Linear PDEs

## *Partial Differential Equations*

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# Objectives

In this lesson we will learn:

- ▶ to classify first-order partial differential equations as either linear or quasilinear,
- ▶ to solve linear first-order partial differential equations,

# Transport Equations

The general form of a first-order scalar PDE is

$$u_x + \nabla \cdot f(x, y, u, \nabla u) = g(x, y, u)$$

where

- ▶  $\nabla \cdot f$  denotes the divergence of  $f$ , and
- ▶  $\nabla u$  denotes the gradient of  $u$ .

# Classification

A first-order PDE is **linear** if it can be written as

$$a(x, y)u_x(x, y) + b(x, y)u_y(x, y) = c(x, y)u(x, y) + d(x, y).$$

A first-order PDE is **semilinear** if it can be written as

$$a(x, y)u_x(x, y) + b(x, y)u_y(x, y) = c(x, y, u).$$

A first-order PDE is **quasilinear** if it can be written as

$$a(x, y, u)u_x(x, y) + b(x, y, u)u_y(x, y) = c(x, y, u).$$

# Simple Case

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$$u_x + c u_y = 0$$

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$$\langle u_x, u_y \rangle \cdot \langle 1, c \rangle = 0$$

$$D_{\langle 1, c \rangle} u(x, y) = 0.$$

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- ▶ This can be interpreted as stating the directional derivative of  $u(x, y)$  in the direction of vector  $\langle 1, c \rangle$  is 0.

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- ▶ Function  $u(x, y)$  is constant along lines of the form  $y - cx = k$ .
- ▶ Function  $u(x, y) \equiv f(k)$  where  $f$  is an arbitrary differentiable function.

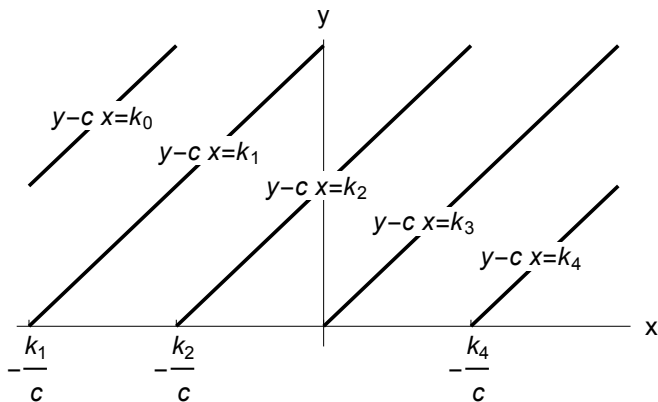


# Illustration

The solutions of

$$u_x + c u_y = 0$$

must be constant along lines of the form  $y - c x = k$ .



# Confirmation of Solution

Let  $u(x, y) = f(y - cx)$  then

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Functions of the form  $u(x, y) = f(y - cx)$  are referred to as the **general solutions** to this simple PDE.

# Initial Conditions

If the value of  $u(x, y)$  along the line where  $x = 0$  is specified, then an initial condition of the form  $u(0, y) = \phi(y)$  further specifies the solution.

$$u(x, y) = f(y - cx) \text{ (general solution)}$$

$$u(0, y) = f(y) = \phi(y) \text{ (initial condition)}$$

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$$u(0, y) = f(y) = \phi(y) \text{ (initial condition)}$$

$$u(x, y) = \phi(y - cx) \text{ (solution to IVP)}$$

# General First-Order Linear PDE

The solution technique used previously can be extended to the **general first-order linear PDE** of the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y)u + d(x, y),$$

for  $(x, y) \in D \subset \mathbb{R}^2$ .

## Vector Field (1 of 3)

Suppose  $(x, y) \equiv (x(t), y(t))$  where  $t$  is a parameter, then

$$\frac{d}{dt} [u(x, y)] = u_x x'(t) + u_y y'(t).$$

Consider

$$a(x, y)u_x + b(x, y)u_y = c(x, y)u + d(x, y) \text{ as}$$

$$\frac{dx}{dt} u_x + \frac{dy}{dt} u_y = c(x, y)u + d(x, y) \text{ where}$$

$$\frac{dx}{dt} = a(x, y)$$

$$\frac{dy}{dt} = b(x, y).$$



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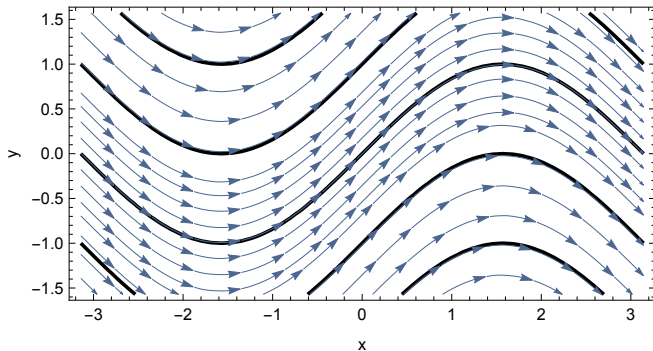
$$\frac{dx}{dt} = a(x, y)$$

$$\frac{dy}{dt} = b(x, y).$$

**Remark:** the parametric curve  $(x(t), y(t))$  is an **integral curve** of the **vector field**  $\langle a(x, y), b(x, y) \rangle$  for all  $(x, y) \in D$ .

## Vector Field (2 of 3)

A vector field and integral curves:



## Vector Field (3 of 3)

If  $\frac{dx}{dt} = a(x, y)$  and  $\frac{dy}{dt} = b(x, y)$  then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b(x, y)}{a(x, y)}.$$

Suppose the implicit form of the solution of this ODE is  $\phi(x, y) = k$  where  $k$  is an arbitrary constant.

Along the curves defined by  $\phi(x, y) = k$  the general first-order linear PDE

$$a(x, y)u_x + b(x, y)u_y = c(x, y)u + d(x, y)$$

becomes

$$\begin{aligned}u_x + u_y \frac{b(x, y)}{a(x, y)} &= \frac{c(x, y)u + d(x, y)}{a(x, y)} \\u_x + u_y \frac{dy}{dx} &= \frac{c(x, y)u + d(x, y)}{a(x, y)} \\ \frac{du}{dx} &= \frac{c(x, y)u + d(x, y)}{a(x, y)}.\end{aligned}$$

# Ordinary Differential Equation

$$\frac{du}{dx} = \frac{c(x, y)u + d(x, y)}{a(x, y)}$$

is an ordinary differential equation for  $u$  as a function of variable  $x$  (since  $y \equiv y(x)$ ) and arbitrary constant  $k$ .

Solving this ODE yields  $u \equiv u(x, k) \equiv u(x, y)$  since  $k = \phi(x, y)$ .

# Characteristics

- ▶ The ordinary differential equations:

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}$$

$$\frac{du}{dt} = c(x, y)u + d(x, y)$$

are called the **characteristic equations**.

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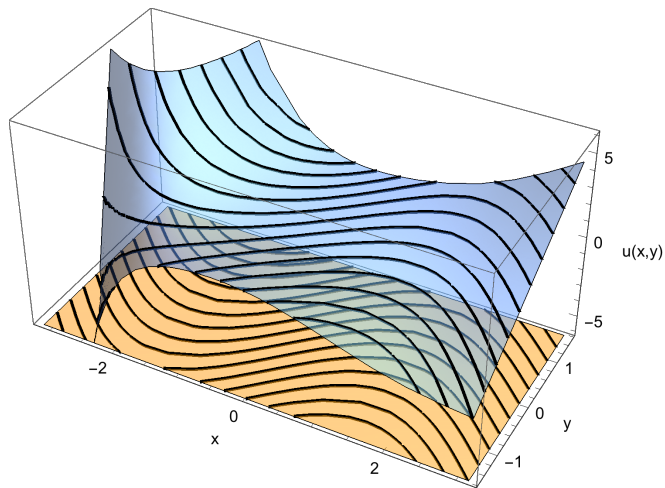
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- ▶ The solution curves  $(x(t), y(t), u(x(t), y(t)))$  to the characteristic equations are called the **characteristic curves**.
- ▶ The curves in the  $xy$ -plane of the form  $(x(t), y(t))$  are called **characteristics**.

# Characteristic Curves vs. Characteristics



Characteristics are curves in the  $xy$ -plane and characteristic curves are curves on the surface  $u(x,y)$ .



# Example

Consider the initial value problem:

$$\begin{aligned}2u_x + 3u_y - 4u &= 0 \\ u(x, 0) &= \sin x.\end{aligned}$$

1. Find the general solution to the PDE using the method of characteristics.
2. Find the solution to the IVP.

## Solution (1 of 2)

$$2u_x + 3u_y - 4u = 0$$

In this PDE we can let  $a(x, y) = 2$  and  $b(x, y) = 3$ , thus

$$\frac{dy}{dx} = \frac{3}{2} \implies y = \frac{3}{2}x + \hat{k} \iff 2y - 3x = k.$$

**Comment:** the characteristics for this PDE are straight lines of the form  $2y - 3x = k$ .

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Think of  $u(x, y) = u(x, y(x))$  so that

$$\frac{du}{dx} = u_x + u_y \frac{dy}{dx} = u_x + \frac{3}{2}u_y = 2u$$

which implies

$$u(x, y) = f(k)e^{2x} = f(2y - 3x)e^{2x}$$

where  $f$  is an arbitrary differentiable function.

## Solution (2 of 2)

In order to satisfy the initial condition:

$$u(x, 0) = \sin x$$

$$f(-3x)e^{2x} = \sin x$$

$$f(-3x) = e^{-2x} \sin x$$

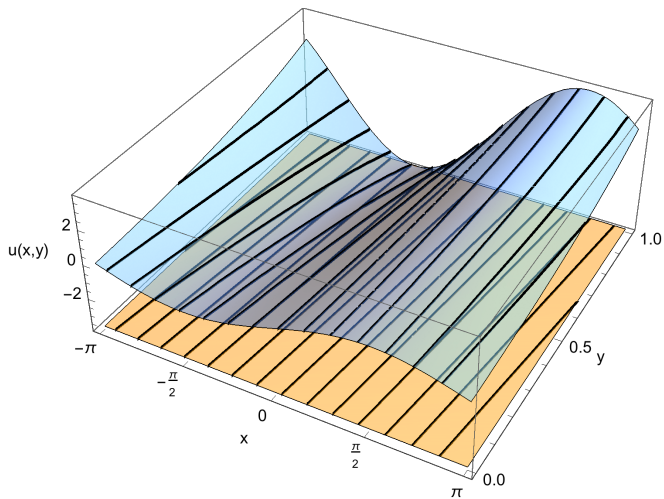
$$f(z) = -e^{2z/3} \sin(z/3).$$

Thus the solution to the IVP is

$$u(x, y) = -e^{2(2y-3x)/3} \sin((2y-3x)/3)e^{2x} = -e^{4y/3} \sin\left(\frac{2}{3}y - x\right).$$

# Illustration

$$u(x, y) = -e^{4y/3} \sin\left(\frac{2}{3}y - x\right)$$



## Example

Find the general solution to the first-order linear PDE:

$$-y u_x + x u_y = 0.$$

## Solution (1 of 2)

If we let  $a(x, y) = -y$  and  $b(x, y) = x$  then the characteristics satisfy the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y \, dy = -x \, dx$$

$$\int y \, dy = - \int x \, dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + \hat{k}$$

$$x^2 + y^2 = k.$$

In this example the characteristics are circles of radius  $\sqrt{k}$  for  $k \geq 0$ .

## Solution (2 of 2)

Assuming  $u(x, y) = u(x, y(x))$  then

$$\begin{aligned}\frac{du}{dx} &= u_x + u_y \frac{dy}{dx} \\ &= u_x - \frac{x}{y} u_y\end{aligned}$$

$$\frac{du}{dx} = 0$$

$$u = f(k)$$

$$u(x, y) = f(x^2 + y^2)$$

where  $f$  is an arbitrary differentiable function.



# Example

Solve the initial value problem:

$$2xy u_x + u_y - u = 0$$

$$u(x, 0) = x,$$

for  $x > 0$  and  $y > 0$ .

## Solution (1 of 2)

Let  $a(x, y) = 2x y$  and  $b(x, y) = 1$  then

$$\frac{dy}{dx} = \frac{1}{2xy} \text{ (separable ODE)}$$

$$y \, dy = \frac{1}{2x} \, dx$$

$$\int y \, dy = \int \frac{1}{2x} \, dx$$

$$\frac{1}{2} y^2 = \frac{1}{2} \ln x + \hat{k}$$

$$y^2 - \ln x = k.$$

Thus curves of the form  $y = \sqrt{k + \ln x}$  are the characteristics of the solution.

## Solution (2 of 2)

To find the general solution to the PDE:

$$2xy u_x + u_y - u = 0$$

$$u_x + \frac{1}{2xy} u_y = \frac{u}{2xy}$$

$$\frac{du}{dx} = \frac{u}{2x\sqrt{k + \ln x}} \quad (\text{separable ODE})$$

$$\frac{1}{u} du = \frac{1}{2x\sqrt{k + \ln x}} dx$$

$$\int \frac{1}{u} du = \int \frac{1}{2x\sqrt{k + \ln x}} dx \quad (\text{substitute } v = k + \ln x)$$

$$\ln u = \sqrt{k + \ln x} + \ln f(k)$$

$$u(x, y) = f(k)e^{\sqrt{k + \ln x}} = f(y^2 - \ln x)e^y$$

where  $f$  is an arbitrary differentiable function.

## Solution (2 of 2)

To find the general solution to the PDE:

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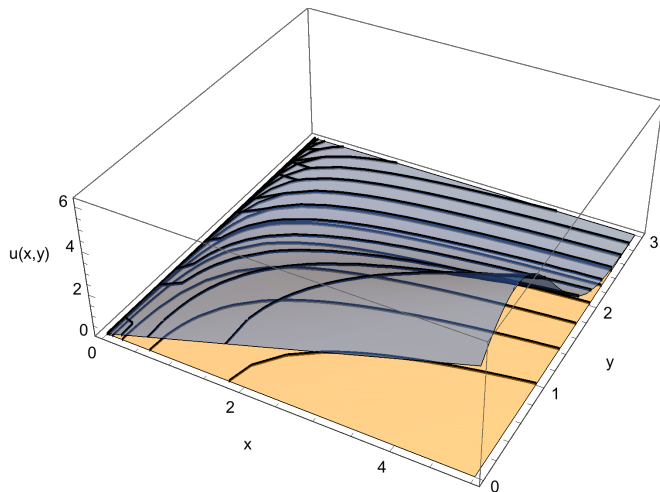
where  $f$  is an arbitrary differentiable function.

Since  $u(x, 0) = x$  then

$$x = f(-\ln x) \iff f(x) = e^{-x} \implies u(x, y) = e^{-y^2 + \ln x} e^y = x e^{y - y^2}.$$

# Illustration

$$u(x, y) = x e^{y-y^2}$$



# Example

Show that the problem

$$\begin{aligned} -y u_x + x u_y &= 0 \\ u(x, 0) &= 3x \end{aligned}$$

has no solution.

# Justification

From a previous example we know the characteristics are the curves of the form  $x^2 + y^2 = k$  and the general solution has the form

$$u(x, y) = f(x^2 + y^2)$$

where  $f$  is an arbitrary differentiable function.

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From a previous example we know the characteristics are the curves of the form  $x^2 + y^2 = k$  and the general solution has the form

$$u(x, y) = f(x^2 + y^2)$$

where  $f$  is an arbitrary differentiable function.

If  $x \neq 0$  then

$$u(-x, 0) = -3x \neq 3x = u(x, 0)$$

but

$$u(-x, 0) = f((-x)^2) = f(x^2) = u(x, 0)$$

which is a contradiction.



# Characteristics

The side condition is specified on the  $x$ -axis which intersects each characteristic curve twice (at  $x = -\sqrt{k}$  and  $x = \sqrt{k}$ ). The side condition specifies two different values of the solution, but the solution must be constant on each characteristic curve.

# Example

Consider the first-order, linear PDE

$$u_x + (\cos x)u_y + u = x y.$$

Find the general solution to this PDE.

## Solution (1 of 4)

Let  $a(x, y) = 1$  and  $b(x, y) = \cos x$ , then the characteristics satisfy the ODE

$$\frac{dy}{dx} = \cos x$$

$$dy = \cos x \, dx$$

$$\int 1 \, dy = \int \cos x \, dx$$

$$y = \sin x + k.$$

In this example the characteristics are curves of the form  $y - \sin x = k$ .

## Solution (2 of 4)

Assuming  $u(x, y) = u(x, y(x))$  then

$$u_x + (\cos x)u_y + u = x y$$

$$\frac{du}{dx} = x y - u = x(\sin x + k) - u$$

$$\frac{du}{dx} + u = x \sin x + k x.$$

This is a first-order, linear ODE which can be solved by multiplying both sides by the integrating factor  $e^x$  and integrating.

## Solution (3 of 5)

$$\frac{du}{dx} + u = x \sin x + k x$$

$$\frac{d}{dx} [u e^x] = x e^x \sin x + k x e^x$$

$$\int d[u e^x] = \int (x e^x \sin x + k x e^x) dx$$

Evaluation of the integral will require integration by parts.

## Solution (4 of 5)

Integrate by parts.

$$\begin{aligned}\int x e^x \sin x \, dx &= -x e^x \cos x + \int (1+x) e^x \cos x \, dx \\ &= -x e^x \cos x + (1+x) e^x \sin x - \int (2+x) e^x \sin x \, dx\end{aligned}$$

$$2 \int x e^x \sin x \, dx = -x e^x \cos x + (1+x) e^x \sin x - 2 \int e^x \sin x \, dx$$

The latter integral can also be handled using integration by parts.

$$\begin{aligned}\int e^x \sin x \, dx &= -e^x \cos x + \int e^x \cos x \, dx \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx\end{aligned}$$

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

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$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

Therefore

$$\int x e^x \sin x \, dx = -\frac{x}{2} e^x \cos x + \frac{1+x}{2} e^x \sin x + \frac{1}{2} e^x (\cos x \sin x).$$

## Solution (5 of 5)

Integrate by parts.

$$\int kxe^x dx = kxe^x - \int ke^x dx = kxe^x - ke^x = k(x-1)e^x$$

Putting it all together.

$$\int d[u e^x] = \int (x e^x \sin x + k x e^x) dx$$

$$u e^x = k(x-1)e^x - \frac{x}{2}e^x \cos x + \frac{1+x}{2}e^x \sin x + \frac{1}{2}e^x(\cos x - \sin x) + f(k)$$

$$u = k(x-1) + \frac{1-x}{2} \cos x + \frac{x}{2} \sin x + e^{-x} f(k)$$

$$u(x, y) = (y - \sin x)(x-1) + \frac{1-x}{2} \cos x + \frac{x}{2} \sin x + e^{-x} f(y - \sin x)$$



# Example

Consider the first-order, linear PDE and side condition

$$y u_x - 4x u_y = 2x y$$
$$u(x, 0) = x^4$$

Find a particular solution to this PDE which satisfies the side condition.

## Solution (1 of 3)

Let  $a(x, y) = y$  and  $b(x, y) = -4x$ , then the characteristics satisfy the ODE

$$\frac{dy}{dx} = -\frac{4x}{y} \text{ (separable ODE)}$$

$$y \, dy = -4x \, dx$$

$$\int y \, dy = - \int 4x \, dx$$

$$\frac{1}{2}y^2 = -2x^2 + \hat{k}.$$

In this example the characteristics are curves of the form  $y^2 + 4x^2 = k$ .

## Solution (2 of 3)

Assuming  $u(x, y) = u(x, y(x))$  then

$$y u_x - 4x u_y = 2x y$$

$$u_x - \frac{4x}{y} u_y = 2x$$

$$\frac{du}{dx} = 2x \text{ (separable ODE)}$$

$$du = 2x dx$$

$$\int 1 du = \int 2x dx$$

$$u = x^2 + f(k)$$

$$u(x, y) = x^2 + f(4x^2 + y^2)$$

where  $f$  is an arbitrary differentiable function.

## Solution (3 of 3)

To satisfy the side condition we need

$$u(x, 0) = x^2 + f(4x^2)$$

$$x^4 = x^2 + f(4x^2)$$

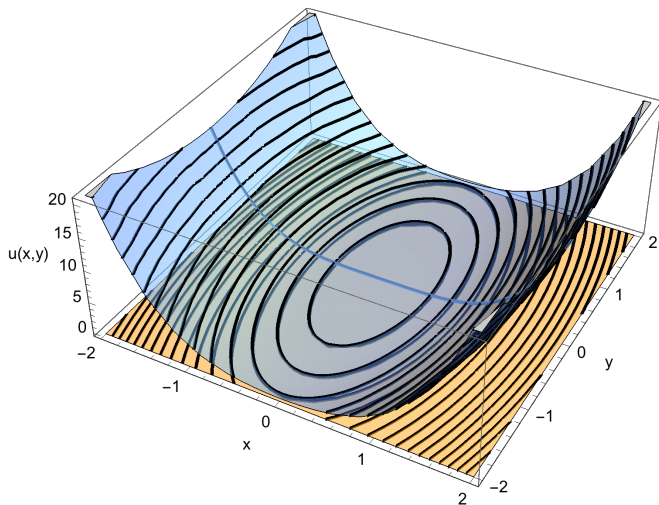
$$f(4x^2) = x^4 - x^2$$

$$f(z) = \frac{z^2}{16} - \frac{z}{4}$$

$$u(x, y) = x^2 + \frac{(4x^2 + y^2)^2}{16} - \frac{4x^2 + y^2}{4}$$

# Illustration

$$u(x, y) = x^2 + \frac{(4x^2 + y^2)^2}{16} - \frac{4x^2 + y^2}{4}$$



# Example

Consider the first-order, linear PDE and side condition

$$y u_x - 4x u_y = 2x y$$

$$u(x, 0) = x^3$$

Show there is no solution to this PDE which satisfies the side condition.

# Justification

From the work done previously we know the general solution to the PDE takes the form

$$u(x, y) = x^2 + f(4x^2 + y^2)$$

where  $f$  is an arbitrary differentiable function.

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From the work done previously we know the general solution to the PDE takes the form

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where  $f$  is an arbitrary differentiable function.

Suppose  $u(x, y)$  is such that  $u(x, 0) = x^3$ , then for any  $x$

$$u(-x, 0) = (-x)^2 + f(4(-x)^2 + 0^2) = x^2 + f(4x^2 + 0^2) = u(x, 0)$$

However, when  $x \neq 0$ ,

$$u(-x, 0) = (-x)^3 \neq x^3 = u(x, 0)$$

which is a contradiction.