First-Order PDEs

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn:

- solutions to linear and quasilinear first-order partial differential equations,
- applications of first-order PDEs as mathematical models of traffic flow, structured populations, *etc*.

Transport Equations

The general form of a first-order scalar PDE is

$$u_t + \nabla \cdot f(x, t, u, \nabla u) = g(x, t, u)$$

where

- $\nabla \cdot f$ denotes the divergence of f, and
- ∇u denotes the gradient of u.

Simple Case

Let c be a constant and consider

$$\begin{aligned} u_t + c\, u_x &= 0 \\ \langle u_x, u_t \rangle \cdot \langle c, 1 \rangle &= 0 \\ D_{\langle c,1 \rangle} u(x,t) &= 0. \end{aligned}$$

Remarks:

- This can be interpreted as stating the directional derivative of $u(x,t)$ in the direction of vector $\langle c, 1 \rangle$ is 0.
- Function $u(x,t)$ is constant along lines of the form $x - c\,t = k$.
- Function $u(x,t) \equiv f(k)$ where f is an arbitrary differentiable function.

Confirmation

Let $u(x, t) = f(x - c\,t)$ then

$$\begin{aligned} u_t &= -c\,f'(x - c\,t) \\ u_x &= f'(x - c\,t) \end{aligned}$$

and

$$u_t + c\,u_x = -c\,f'(x - c\,t) + c\,f'(x - c\,t) = 0.$$

Functions of the form $u(x, t) = f(x - c\,t)$ are referred to as the general solution to this simple PDE.

Initial Conditions

If the value of $u(x,t)$ along the line where $t = 0$ is specified, then an initial condition of the form $u(x,0) = \phi(x)$ further specifies the solution.

$$
\begin{aligned}
u(x,t) &= f(x - c\,t) && \text{(general solution)} \\
u(x,0) &= f(x) = \phi(x) && \text{(initial condition)} \\
u(x,t) &= \phi(x - c\,t) && \text{(solution to IVP)}
\end{aligned}
$$

General First-Order Linear PDE

The solution technique used previously can be extended to the **general first-order linear PDE** of the form

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y),$$

for $(x,y) \in D \subset \mathbb{R}^2$.

Note: symbol y is used in place of t as in the previous example. This change of variable is made so that t can be used as a parameter in the solution procedure to be outlined next.

Vector Field (1 of 2)

Suppose $(x, y) \equiv (x(t), y(t))$ where t is a parameter, then

$$\frac{d}{dt} u(x, y) = u_x x'(t) + u_y y'(t)$$

and we can think of

$$\begin{aligned} a(x, y)u_x + b(x, y)u_y + c(x, y)u &= d(x, y) \quad \text{as} \\ u_x x'(t) + u_y y'(t) &= -c(x, y)u + d(x, y) \quad \text{where} \\ \frac{dx}{dt} &= a(x, y) \\ \frac{dy}{dt} &= b(x, y). \end{aligned}$$

Remark: the parametric curve $(x(t), y(t))$ is tangent to the vector field $\langle a(x, y), b(x, y) \rangle$ for all $(x, y) \in D$.

Vector Field (2 of 2)

If $\dfrac{dx}{dt} = a(x,y)$ and $\dfrac{dy}{dt} = b(x,y)$ then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b(x,y)}{a(x,y)}.$$

Suppose the implicit form of the solution of this ODE is $\phi(x,y) = k$ where k is an arbitrary constant.

Along the curves defined by $\phi(x,y) = k$ the general first-order linear PDE

$$a(x,y)u_x + b(x,y)u_y = -c(x,y)u + d(x,y)$$

becomes

$$\begin{aligned} u_x + u_y \frac{b(x,y)}{a(x,y)} &= \frac{-c(x,y)u + d(x,y)}{a(x,y)} \\ u_x + u_y \frac{dy}{dx} &= \frac{-c(x,y)u + d(x,y)}{a(x,y)} \\ \frac{du}{dx} &= \frac{-c(x,y)u + d(x,y)}{a(x,y)}. \end{aligned}$$

Ordinary Differential Equation

$$\frac{du}{dx} = \frac{-c(x,y)u + d(x,y)}{a(x,y)}$$

is an ordinary differential equation for u as a function of variable x (since $y \equiv y(x)$) and arbitrary constant k.

Solving this ODE yields $u \equiv u(x,k) \equiv u(x,y)$ since $k = \phi(x,y)$.

Characteristics

- The ordinary differential equations

$$\begin{aligned} \frac{dy}{dx} &= \frac{b(x,y)}{a(x,y)} \\ \frac{du}{dt} &= -c(x,y)u + d(x,y) \end{aligned}$$

  are called the **characteristic equations.**
- The solution curves $(x(t), y(t), u(x(t), y(t)))$ to the characteristic equations are called the **characteristic curves.**
- The curves in the *xy*-plane of the form $(x(t), y(t))$ are called **characteristics.**

Example

Consider the initial value problem:

$$\begin{aligned} 2u_x + 3u_y - 4u &= 0 \\ u(x,0) &= \sin x. \end{aligned}$$

1. Find the general solution to the PDE using the method of characteristics.
2. Find the solution to the IVP.

Solution (1 of 2)

$$2u_x + 3u_y - 4u = 0$$

In this PDE we can let $a(x, y) = 2$ and $b(x, y) = 3$, thus

$$\frac{dy}{dx} = \frac{3}{2} \implies y = \frac{3}{2}x + \hat{k} \iff 2y - 3x = k.$$

Comment: the characteristics for this PDE are straight lines of the form $2y - 3x = k$.

Think of $u(x, y) = u(x, y(x))$ so that

$$\frac{du}{dx} = u_x + u_y \frac{dy}{dx} = u_x + \frac{3}{2}u_y = 2u$$

which implies

$$u(x, y) = f(k)e^{2x} = f(2y - 3x)e^{2x}$$

where f is an arbitrary differentiable function.

Solution (2 of 2)

In order to satisfy the initial condition

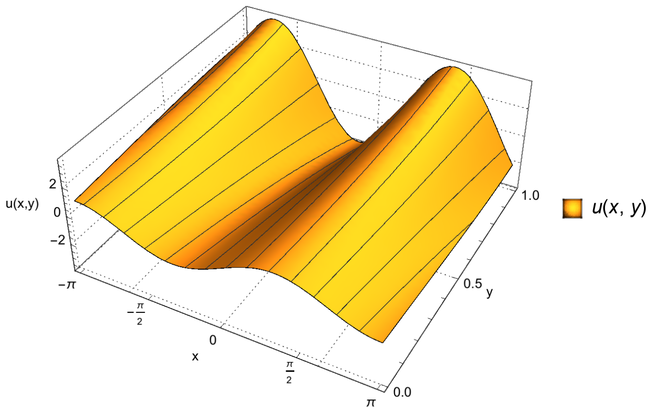
$$\begin{aligned} u(x,0) &= \sin x \\ f(-3x)e^{2x} &= \sin x \\ f(-3x) &= e^{-2x}\sin x \\ f(z) &= -e^{2z/3}\sin(z/3). \end{aligned}$$

Thus the solution to the IVP is

$$u(x,y) = -e^{2(2y-3x)/3}\sin((2y-3x)/2)e^{2x} = -e^{4y/3}\sin\left(y - \frac{3}{2}x\right).$$

Illustration

$$u(x, y) = -e^{4y/3} \sin\left(y - \frac{3}{2}x\right)$$



Example

Find the general solution to the first-order linear PDE:

$$-y\, u_x + x\, u_y = 0.$$

Example

Solve the initial value problem:

$$\begin{aligned} 2x\,y\,u_x + u_y - u &= 0 \\ u(x,0) &= x, \end{aligned}$$

for $x > 0$ and $y > 0$.

Homework

- Read Section 2.1
- Exercises: 1–5

Quasilinear and Semilinear PDEs

A first-order PDE is called **quasilinear** if it has the form

$$a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0$$

(assuming $u \equiv u(x, y)$ for $(x, y) \in D \subset \mathbb{R}^2$).

A first-order PDE is called **semilinear** if it has the form

$$a(x, y)u_x + b(x, y)u_y - c(x, y, u) = 0.$$

We will generalize the method of characteristics in order to solve quasilinear PDEs.

Method of Lagrange

Solutions of the quasilinear PDE

$$a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0$$

can be expressed implicitly as $\varphi(x, y, z) = 0$ when $\varphi(x, y, z)$ is a solution to the PDE

$$a(x, y, z)\varphi_x + b(x, y, z)\varphi_y + c(x, y, z)\varphi_z = 0.$$

Proof (1 of 3)

- Suppose $u \equiv u(x, y)$ is a solution to

$$a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0.$$

- Define $\varphi(x, y, z) = u(x, y) - z$, then

$$\varphi_x = u_x, \quad \varphi_y = u_y, \quad \varphi_z = -1.$$

- Substitute into the expression

$$\begin{aligned} & a(x, y, z)\varphi_x + b(x, y, z)\varphi_y + c(x, y, z)\varphi_z \\ = \quad & a(x, y, u(x, y))u_x + b(x, y, u(x, y))u_y + c(x, y, u(x, y))(-1) \\ = \quad & 0, \end{aligned}$$

when $z = u(x, y)$ or equivalently when $\varphi(x, y, z) = 0$.

Proof (2 of 3)

- Suppose $\varphi(x, y, z)$ is a solution of

$$a(x, y, z)\varphi_x + b(x, y, z)\varphi_y + c(x, y, z)\varphi_z = 0$$

  and let (x_0, y_0, z_0) be a point on the surface $\varphi(x, y, z) = 0$ for which $\varphi_z(x_0, y_0, z_0) \neq 0$.
- Near point (x_0, y_0, z_0) the surface $\varphi(x, y, z) = 0$ is the graph of some function, call it $u(x, y)$.

$$\varphi(x, y, u(x, y)) = 0$$

- Differentiate the equation above.

$$\begin{aligned} \varphi_x + \varphi_z u_x &= 0 \iff \varphi_x = -\varphi_z u_x \\ \varphi_y + \varphi_z u_y &= 0 \iff \varphi_y = -\varphi_z u_y \end{aligned}$$

Proof (3 of 3)

- So far we have $\varphi(x, y, z) = 0$ when $z = u(x, y)$ where $\varphi(x, y, z)$ is a solution of

$$a(x, y, z)\varphi_x + b(x, y, z)\varphi_y + c(x, y, z)\varphi_z = 0$$

- We also know $\varphi_x = -\varphi_z u_x$ and $\varphi_y = -\varphi_z u_y$ where $\varphi_z \neq 0$.
- Substitute into the PDE above.

$$\begin{aligned} -a(x, y, u)\varphi_z u_x - b(x, y, u)\varphi_z u_y + c(x, y, z)\varphi_z &= 0 \\ a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) &= 0 \end{aligned}$$

So $u(x, y)$ solves the quasilinear PDE.

Integral Surface

Suppose $u(x, y)$ solves the quasilinear PDE:

$$a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0$$

for $(x, y) \in D$, then the surface

$$S = \{(x, y, u(x, y)) \,:\, (x, y) \in D\}$$

is called an **integral surface.** The vector $\langle u_x, u_y, -1 \rangle$ is normal to the integral surface for all $(x, y) \in D$.

Since

$$\begin{aligned} a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) &= 0 \\ \langle u_x, u_y, -1 \rangle \cdot \langle a(x, y, u), b(x, y, u), c(x, y, u) \rangle &= 0 \end{aligned}$$

then vector $\langle a(x, y, u), b(x, y, u), c(x, y, u) \rangle$ is perpendicular to the normal to the integral surface, *i.e.*, tangent to the integral surface.

Characteristic System

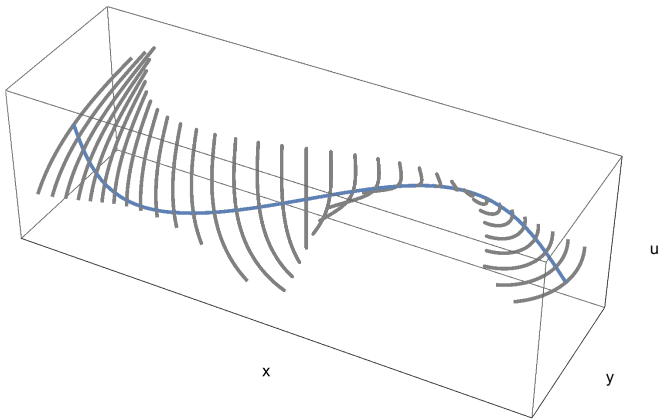
The vector $\langle a(x, y, u), b(x, y, u), c(x, y, u)\rangle$ defines a vector field in xyu-space. The integral curves $(x(t), y(t), u(t))$ defined by the **characteristic system** of ODEs

$$\begin{aligned} \frac{dx}{dt} &= a(x, y, u) \\ \frac{dy}{dt} &= b(x, y, u) \\ \frac{du}{dt} &= c(x, y, u) \end{aligned}$$

are called the **characteristic curves**. The projections of the characteristic curves in the xy-plane will be called **characteristics**.

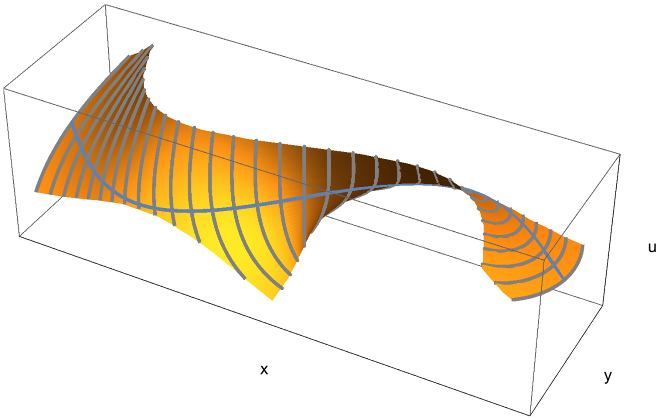
Constructing a Solution

Suppose Γ is a *non-characteristic* curve in *xyu*-space and we construct a family of characteristic curves through points of Γ.



Solution Surface

The union of all the characteristic curves through points on Γ is an solution (integral) surface to the quasilinear PDE.



Example

Find the solution of

$$y\,u_x - x\,u_y - e^u = 0$$

that passes through the curve
$\Gamma = \{(x, y, u) = (s, \sin s, 0) \ : \ s \in \mathbb{R}\}$.

Remarks:

- Γ is parameterized by s for clarity, this parameter will not be used for any other purpose.
- The solution u must satisfy the condition

  $$u(x, \sin x) = 0$$

  for all x.

Solution (1 of 5)

The characteristic system for this example is

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x, \quad \frac{du}{dt} = e^u.$$

From the last equation, $-e^{-u} = t - C$ where C is a constant.

From the second equation,

$$\frac{dy}{dt} = -x = \frac{d}{dt}\left[\frac{dx}{dt}\right] = \frac{d^2x}{dt^2} \iff x''(t) + x = 0.$$

Thus $x(t) = A\cos t + B\sin t$ and consequently
$y(t) = x'(t) = -A\sin t + B\cos t$ where A and B are constants.

Solution (2 of 5)

The characteristic curves of the solution can be parameterized as

$$\begin{aligned} x &= A\cos t + B\sin t \\ y &= B\cos t - A\sin t \\ u &= -\ln(C - t). \end{aligned}$$

The non-characteristic curve Γ is parameterized as

$$\begin{aligned} x &= s \\ y &= \sin s \\ u &= 0. \end{aligned}$$

For each point on Γ we want to find a characteristic curve which passes through the point when $t = 0$ (arbitrary choice).

Solution (3 of 5)

When $t = 0$,

$$\begin{aligned} x(0) &= s = A \\ y(0) &= \sin s = B \\ u(0) &= 0 = -\ln C \iff C = 1 \end{aligned}$$

The characteristic curve intersecting Γ at $t = 0$ can be parameterized as

$$\begin{aligned} x(t) &= s\cos t + \sin s \sin t \\ y(t) &= \sin s \cos t - s \sin t \\ u(t) &= -\ln(1 - t). \end{aligned}$$

These equations also produce the integral surface for the solution of the quasilinear PDE.

Solution (4 of 5)

$$\begin{aligned} x &= s\cos t + \sin s \sin t \\ y &= \sin s \cos t - s \sin t \\ u &= -\ln(1-t). \end{aligned}$$

We can eliminate s and t from the equations. Multiply the first equation by $\cos t$ and the second equation by $-\sin t$ and add them together.

$$\begin{aligned} x\cos t - y \sin t &= s\cos^2 t + \sin s \cos t \sin t - \sin s \cos t \sin t + s \sin^2 t \\ &= s \end{aligned}$$

Multiply the first equation by $\sin t$ and the second equation by $\cos t$ and add them together.

$$\begin{aligned} x\sin t + y \cos t &= s\cos t \sin t + \sin s \sin^2 t + \sin s \cos^2 - s \cos t \sin t \\ &= \sin s \end{aligned}$$

Combining the results produces:

$$\sin(x\cos t - y \sin t) = x \sin t + y \cos t.$$

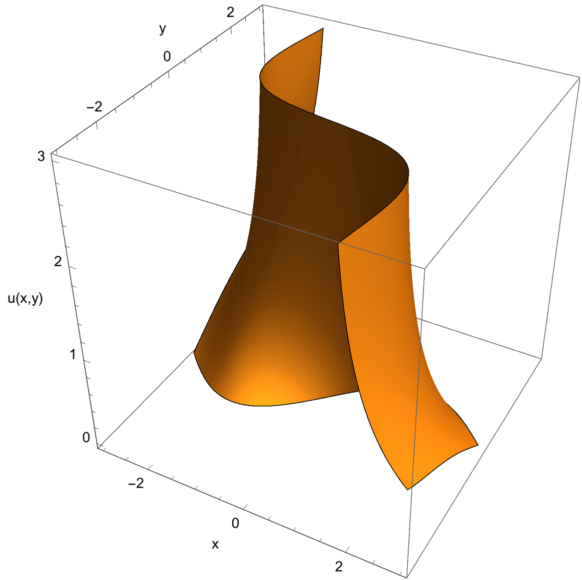
Solution (5 of 5)

We managed to eliminate parameter s. Using $u = -\ln(1-t) \iff t = 1 - e^{-u}$ we can eliminate parameter t as well.

The solution to the quasilinear PDE can be expressed in implicit form as

$$\begin{aligned} & \sin\left(x\cos\left[1-e^{-u}\right] - y\sin\left[1-e^{-u}\right]\right) \\ = \; & x\sin\left[1-e^{-u}\right] + y\cos\left[1-e^{-u}\right]. \end{aligned}$$

Illustration



Example

Consider the quasilinear first-order PDE

$$x\, u_x + y\, u_y - \sec u = 0.$$

Find the solution passing through the parametric curve Γ given by $\Gamma = \{(x, y, u) = (s^2, \sin s, 0) \; : \; s \in \mathbb{R}\}$.

Example

Find the solution to the first-order, quasi linear PDE with side condition.

$$\begin{aligned} u_x + u\,u_y &= 6x \\ u(0,y) &= 3y \end{aligned}$$

Remark: since no non-characteristic curve Γ is mentioned we are free to create one. A natural one to use would be $\Gamma = \{(x,y,u) = (0,s,3s) \,:\, s \in \mathbb{R}\}$.

Solution (1 of 3)

Re-write the PDE as

$$u_x + u\,u_y - 6x = 0$$

which has the characteristic system:

$$\begin{aligned}
\frac{dx}{dt} &= 1 \implies x = t + A \\
\frac{dy}{dt} &= u = 3t^2 + 6A\,t + B \implies y = t^3 + 3A\,t^2 + B\,t + C \\
\frac{du}{dt} &= 6x = 6t + 6A \implies u = 3t^2 + 6A\,t + B.
\end{aligned}$$

Solution (2 of 3)

For each point on Γ find the characteristic curve which passes through the point when $t = 0$ (arbitrary choice).

$$\begin{aligned} x(0) &= 0 = A \\ y(0) &= s = C \\ u(0) &= 3s = B \end{aligned}$$

Consequently the characteristic curves are in parametric form:

$$\begin{aligned} x(t) &= t \\ y(t) &= t^3 + s(3t + 1) \\ u(t) &= 3t^2 + 3s. \end{aligned}$$

Solution (3 of 3)

Now eliminate s and t so as to write the solution in implicit form. Taking the parametric equations for x and u we can solve for s and t to obtain

$$\begin{aligned} s &= \frac{u}{3} - x^2 \\ t &= x. \end{aligned}$$

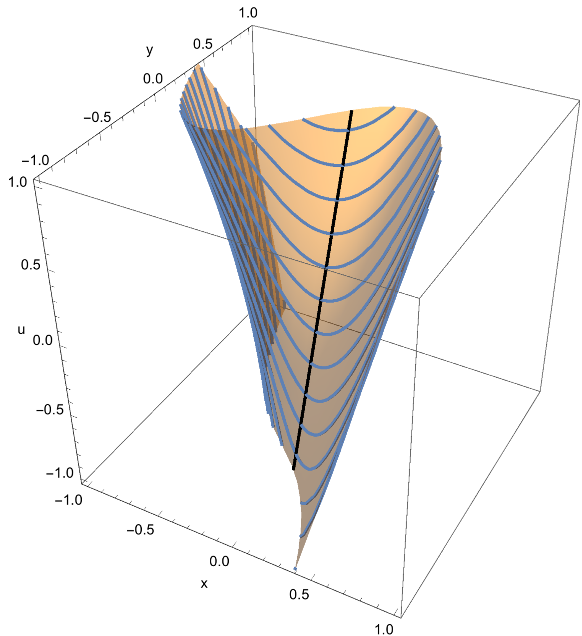
Substituting into the equation for y we get

$$y = \frac{u}{3} + x\,u - x^2 - 3x^3$$

which can be solved explicitly for

$$u(x, y) = \frac{3(2x^3 + x^2 + y)}{3x + 1}.$$

Illustration



Homework

- Read Section 2.2
- Exercises: 6–8