

# First-Order Quasilinear PDEs

MATH 467 *Partial Differential Equations*

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## Objectives

In this lesson we will learn:

- ▶ to solve semilinear and quasilinear first-order partial differential equations.

# Quasilinear and Semilinear PDEs

A first-order PDE is called **quasilinear** if it has the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

(assuming  $u \equiv u(x, y)$  for  $(x, y) \in D \subset \mathbb{R}^2$ ).

A first-order PDE is called **semilinear** if it has the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u).$$

We will generalize the method of characteristics in order to solve quasilinear PDEs (of which semilinear PDEs are a special case).

## Integral Surface

Suppose  $u(x, y)$  solves the quasilinear PDE:

$$a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0$$

for  $(x, y) \in D$ , then the surface

$$S = \{(x, y, u(x, y)) : (x, y) \in D\}$$

is called an **integral surface**. The vector  $\langle u_x, u_y, -1 \rangle$  is normal to the integral surface for all  $(x, y) \in D$ .

Since

$$\begin{aligned} a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) &= 0 \\ \langle u_x, u_y, -1 \rangle \cdot \langle a(x, y, u), b(x, y, u), c(x, y, u) \rangle &= 0 \end{aligned}$$

then vector  $\langle a(x, y, u), b(x, y, u), c(x, y, u) \rangle$  is perpendicular to the normal to the integral surface, *i.e.*, tangent to the integral surface.

# Characteristic System

The vector  $\langle a(x, y, u), b(x, y, u), c(x, y, u) \rangle$  defines a vector field in  $xyu$ -space. The integral curves  $(x(t), y(t), u(t))$  defined by the **characteristic system** of ODEs

$$\frac{dx}{dt} = a(x, y, u)$$

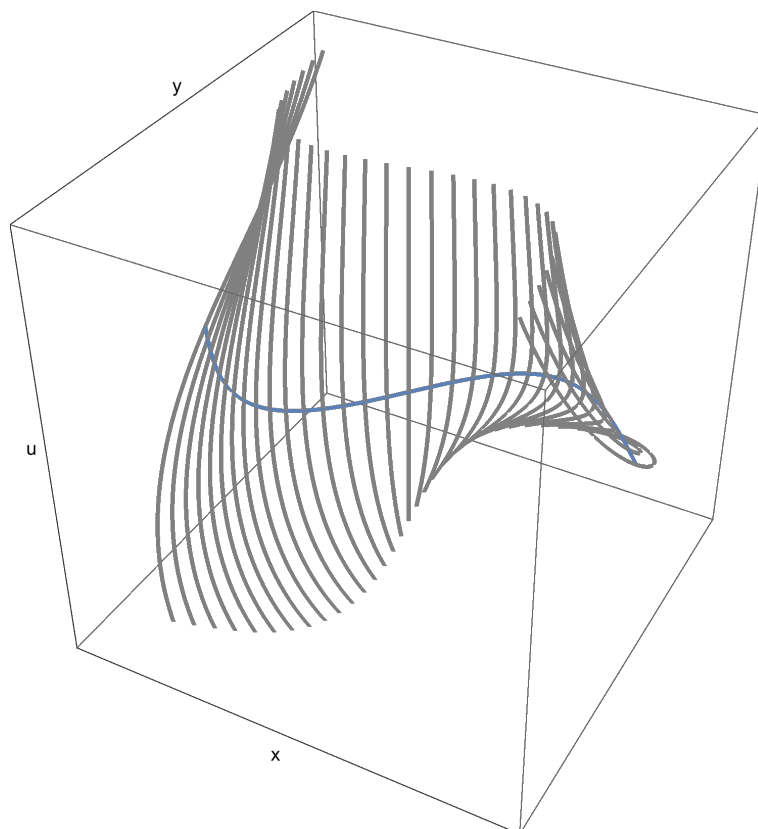
$$\frac{dy}{dt} = b(x, y, u)$$

$$\frac{du}{dt} = c(x, y, u)$$

are called the **characteristic curves**. The projections of the characteristic curves in the  $xy$ -plane will be called **characteristics**.

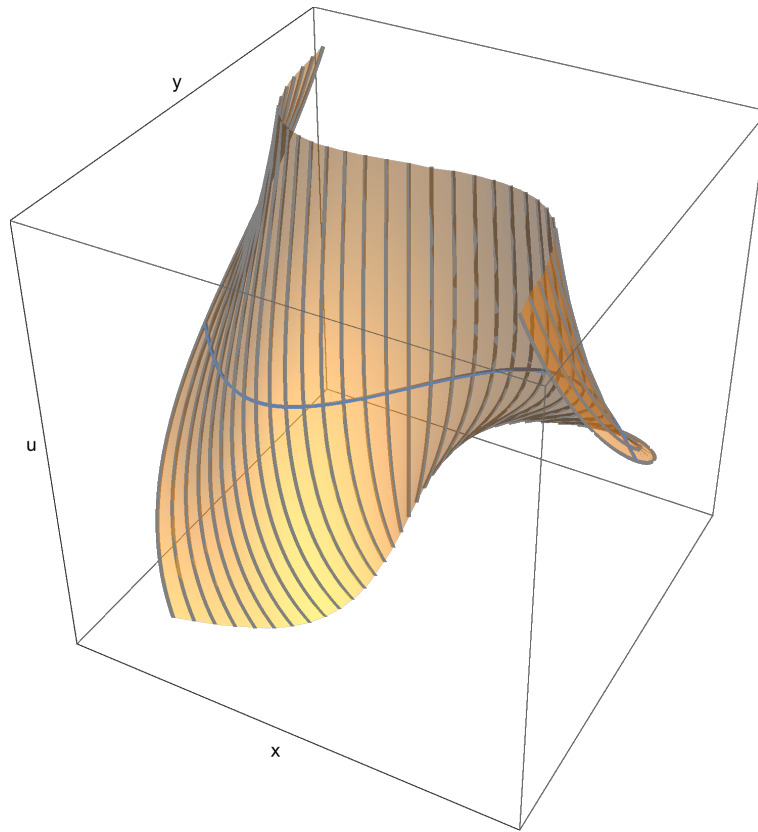
## Constructing a Solution

Suppose  $\Gamma$  is a *non-characteristic* curve in  $xyu$ -space and we construct a family of characteristic curves through points of  $\Gamma$ .



# Solution Surface

The union of all the characteristic curves through points on  $\Gamma$  is a solution (integral) surface to the quasilinear PDE.



## Example

Find the solution of

$$y u_x - x u_y - e^u = 0$$

that passes through the curve  $\Gamma = \{(x, y, u) = (s, \sin s, 0) : s \in \mathbb{R}\}$ .

**Remarks:**

- ▶  $\Gamma$  is parameterized by  $s$  for clarity, this parameter will not be used for any other purpose.
- ▶ The solution  $u$  must satisfy the condition

$$u(x, \sin x) = 0$$

for all  $x$ .

## Solution (1 of 5)

The characteristic system for this example is

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x, \quad \frac{du}{dt} = e^u.$$

From the last equation,  $-e^{-u} = t - C$  where  $C$  is a constant.

From the second equation,

$$\frac{dy}{dt} = -x = \frac{d}{dt} \left[ \frac{dx}{dt} \right] = \frac{d^2x}{dt^2} \iff x''(t) + x = 0.$$

Thus  $x(t) = A \cos t + B \sin t$  and consequently  
 $y(t) = x'(t) = -A \sin t + B \cos t$  where  $A$  and  $B$  are constants.

## Solution (2 of 5)

The characteristic curves of the solution can be parameterized as

$$\begin{aligned}x &= A \cos t + B \sin t \\y &= B \cos t - A \sin t \\u &= -\ln(C - t).\end{aligned}$$

The non-characteristic curve  $\Gamma$  is parameterized as

$$\begin{aligned}x &= s \\y &= \sin s \\u &= 0.\end{aligned}$$

For each point on  $\Gamma$  we want to find a characteristic curve which passes through the point when  $t = 0$  (arbitrary choice).

## Solution (3 of 5)

When  $t = 0$ ,

$$x(0) = s = A$$

$$y(0) = \sin s = B$$

$$u(0) = 0 = -\ln C \iff C = 1$$

The characteristic curve intersecting  $\Gamma$  at  $t = 0$  can be parameterized as

$$x(t) = s \cos t + \sin s \sin t$$

$$y(t) = \sin s \cos t - s \sin t$$

$$u(t) = -\ln(1 - t).$$

These equations also produce the integral surface for the solution of the quasilinear PDE.

## Solution (4 of 5)

$$x = s \cos t + \sin s \sin t$$

$$y = \sin s \cos t - s \sin t$$

$$u = -\ln(1 - t).$$

We can eliminate  $s$  and  $t$  from the equations. Multiply the first equation by  $\cos t$  and the second equation by  $-\sin t$  and add them together.

$$\begin{aligned} x \cos t - y \sin t &= s \cos^2 t + \sin s \cos t \sin t - \sin s \cos t \sin t + s \sin^2 t \\ &= s \end{aligned}$$

Multiply the first equation by  $\sin t$  and the second equation by  $\cos t$  and add them together.

$$\begin{aligned} x \sin t + y \cos t &= s \cos t \sin t + \sin s \sin^2 t + \sin s \cos^2 t - s \cos t \sin t \\ &= \sin s \end{aligned}$$

Combining the results produces:

$$\sin(x \cos t - y \sin t) = x \sin t + y \cos t.$$

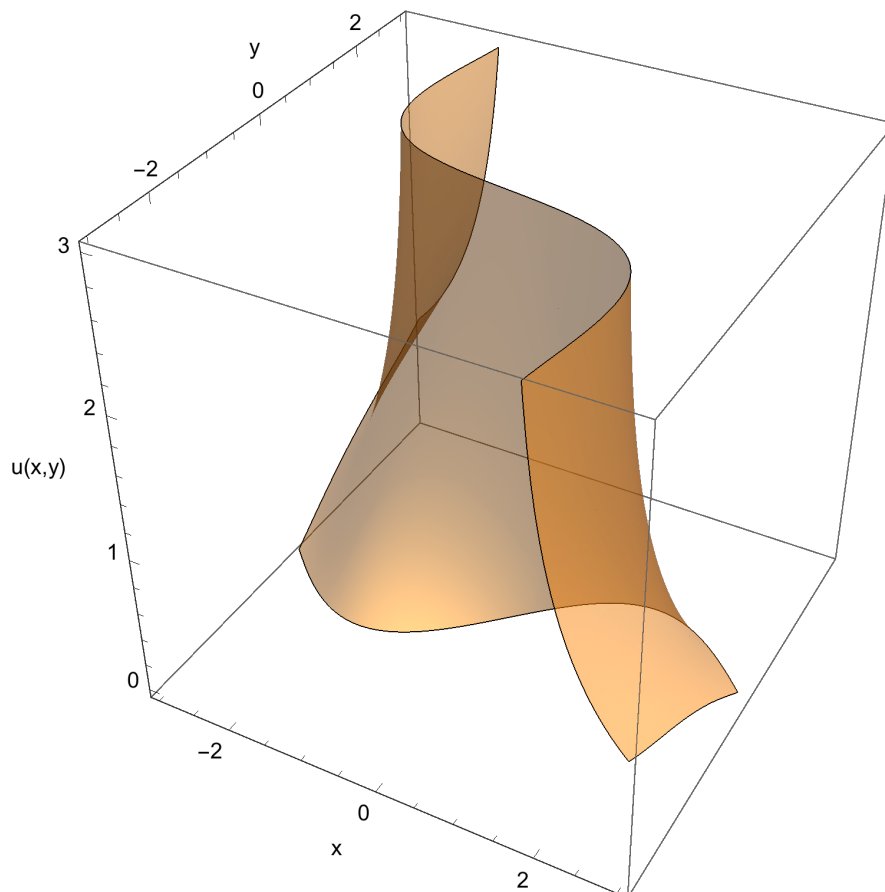
## Solution (5 of 5)

We managed to eliminate parameter  $s$ . Using  $u = -\ln(1 - t) \iff t = 1 - e^{-u}$  we can eliminate parameter  $t$  as well.

The solution to the quasilinear PDE can be expressed in implicit form as

$$\begin{aligned} & \sin(x \cos[1 - e^{-u}] - y \sin[1 - e^{-u}]) \\ &= x \sin[1 - e^{-u}] + y \cos[1 - e^{-u}]. \end{aligned}$$

## Illustration



## Example

Consider the quasilinear first-order PDE

$$x u_x + y u_y - \sec u = 0.$$

Find the solution passing through the parametric curve  $\Gamma$  given by  $\Gamma = \{(x, y, u) = (s^2, \sin s, 0) : s \in \mathbb{R}\}$ .

## Solution (1 of 3)

The characteristic system is:

$$\begin{aligned}\frac{dx}{dt} &= x \implies x(t) = A e^t \\ \frac{dy}{dt} &= y \implies y(t) = B e^t \\ \frac{du}{dt} &= \sec u \implies \sin u = t + C\end{aligned}$$

where  $A$ ,  $B$ , and  $C$  are constants.

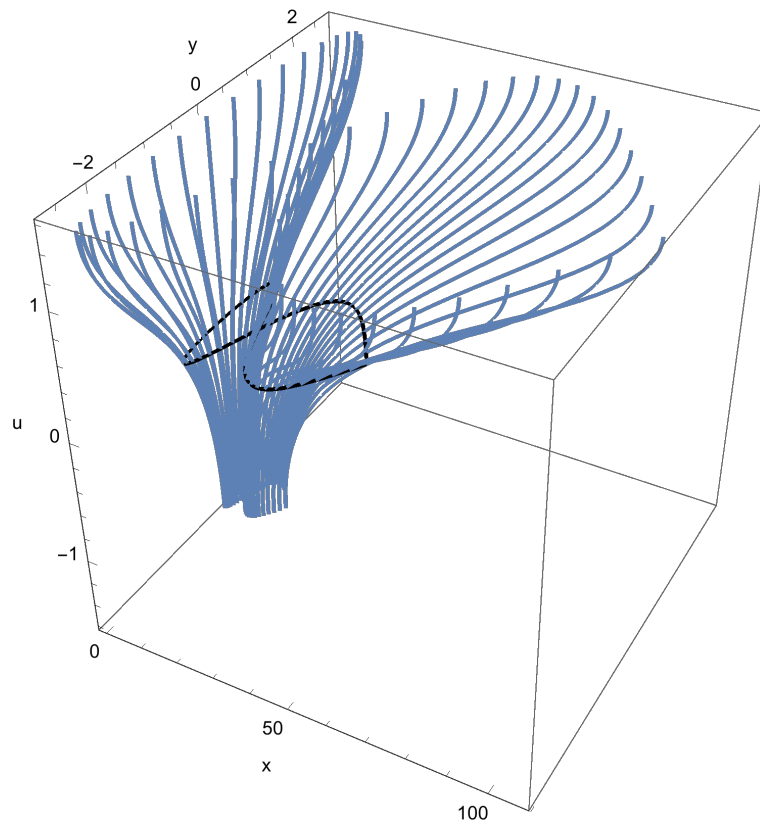
Suppose the characteristics pass through  $\Gamma$  at  $t = 0$  (arbitrary choice).

$$\begin{aligned}x(0) &= A = s^2 \\ y(0) &= B = \sin s \\ u(0) &= 0 = 0 + C \implies C = 0\end{aligned}$$

## Solution (2 of 3)

The characteristic curves take the form:

$$x = s^2 e^t, \quad y = e^t \sin s, \quad u = \arcsin t.$$



## Solution (3 of 3)

We can eliminate parameters  $s$  and  $t$  from the characteristics curves. Since  $t = \sin u$  then

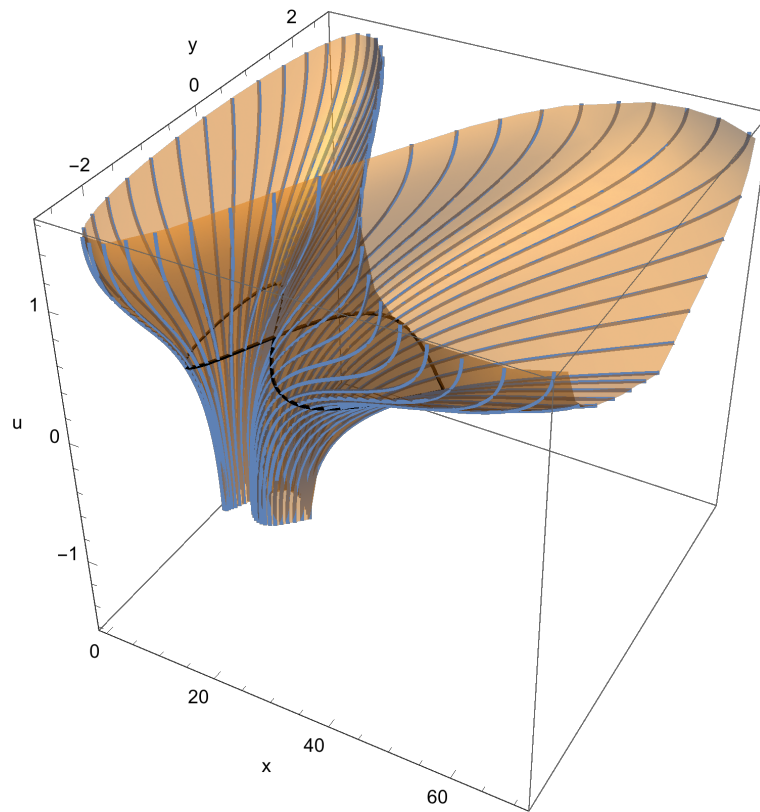
$$y = e^{\sin u} \sin s \iff \sin s = y e^{-\sin u} \implies s = \arcsin (y e^{-\sin u}).$$

Consequently the implicit form of the integral surface can be expressed as

$$x = [\arcsin (y e^{-\sin u})]^2 e^{\sin u}.$$

# Illustration

$$x = \left[ \arcsin \left( y e^{-\sin u} \right) \right]^2 e^{\sin u}$$



## Example

Find the solution to the first-order, quasilinear PDE with side condition.

$$\begin{aligned} u_x + u u_y &= 6x \\ u(0, y) &= 3y \end{aligned}$$

**Remark:** since no non-characteristic curve  $\Gamma$  is mentioned we are free to create one. A natural one to use would be  $\Gamma = \{(x, y, u) = (0, s, 3s) : s \in \mathbb{R}\}$ .

## Solution (1 of 3)

Re-write the PDE as

$$u_x + u u_y - 6x = 0$$

which has the characteristic system:

$$\frac{dx}{dt} = 1 \implies x = t + A$$

$$\frac{dy}{dt} = u = 3t^2 + 6At + B \implies y = t^3 + 3At^2 + Bt + C$$

$$\frac{du}{dt} = 6x = 6t + 6A \implies u = 3t^2 + 6At + B.$$

## Solution (2 of 3)

For each point on  $\Gamma$  find the characteristic curve which passes through the point when  $t = 0$  (arbitrary choice).

$$x(0) = 0 = A$$

$$y(0) = s = C$$

$$u(0) = 3s = B$$

Consequently the characteristic curves are in parametric form:

$$x(t) = t$$

$$y(t) = t^3 + s(3t + 1)$$

$$u(t) = 3t^2 + 3s.$$

## Solution (3 of 3)

Now eliminate  $s$  and  $t$  so as to write the solution in implicit form.  
Taking the parametric equations for  $x$  and  $u$  we can solve for  $s$  and  $t$  to obtain

$$s = \frac{u}{3} - x^2$$
$$t = x.$$

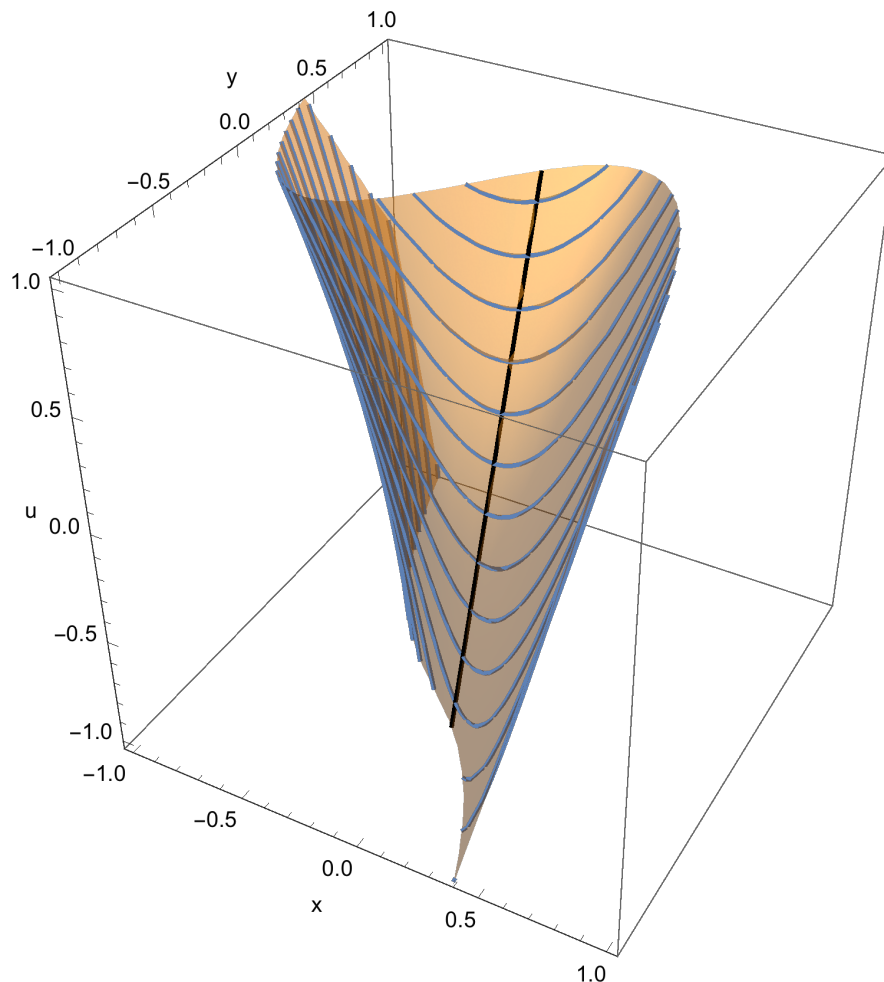
Substituting into the equation for  $y$  we get

$$y = \frac{u}{3} + x u - x^2 - 3x^3$$

which can be solved explicitly for

$$u(x, y) = \frac{3(2x^3 + x^2 + y)}{3x + 1}.$$

## Illustration



# Homework

- ▶ Read Section 2.2
- ▶ Exercises: 6–11