First-Order Quasilinear PDEs

Partial Differential Equations

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Objectives

In this lesson we will learn:

- to solve semilinear and quasilinear first-order partial differential equations.

Quasilinear and Semilinear PDEs

A first-order PDE is called **quasilinear** if it has the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

(assuming $u \equiv u(x, y)$ for $(x, y) \in D \subset \mathbb{R}^2$).

A first-order PDE is called **semilinear** if it has the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u).$$

We will generalize the method of characteristics in order to solve quasilinear PDEs (of which semilinear PDEs are a special case).

Integral Surface

Suppose $u(x, y)$ solves the quasilinear PDE:

$$a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0$$

for $(x, y) \in D$, then the surface

$$S = \{(x, y, u(x, y)) \,:\, (x, y) \in D\}$$

is called an **integral surface**. The vector $\langle u_x, u_y, -1 \rangle$ is normal to the integral surface for all $(x, y) \in D$.

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Since

$$\begin{aligned} a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) &= 0 \\ \langle u_x, u_y, -1\rangle \cdot \langle a(x, y, u), b(x, y, u), c(x, y, u)\rangle &= 0 \end{aligned}$$

then vector $\langle a(x, y, u), b(x, y, u), c(x, y, u)\rangle$ is perpendicular to the normal to the integral surface, *i.e.*, tangent to the integral surface.

Characteristic System

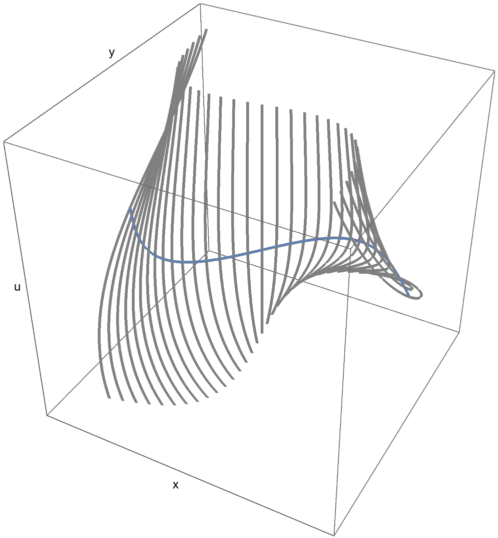
The vector $\langle a(x,y,u), b(x,y,u), c(x,y,u)\rangle$ defines a vector field in xyu-space. The integral curves $(x(t), y(t), u(t))$ defined by the **characteristic system** of ODEs

$$\frac{dx}{dt} = a(x,y,u)$$
$$\frac{dy}{dt} = b(x,y,u)$$
$$\frac{du}{dt} = c(x,y,u)$$

are called the **characteristic curves.** The projections of the characteristic curves in the xy-plane will be called **characteristics.**

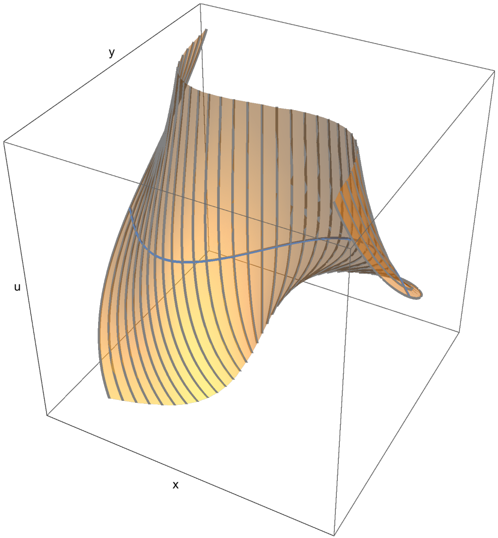
Constructing a Solution

Suppose Γ is a *non-characteristic* curve in *xyu*-space and we construct a family of characteristic curves through points of Γ.



Solution Surface

The union of all the characteristic curves through points on Γ is a solution (integral) surface to the quasilinear PDE.



Example

Find the solution of

$$y\, u_x - x\, u_y - e^u = 0$$

that passes through the curve $\Gamma = \{(x, y, u) = (s, \sin s, 0) \;:\; s \in \mathbb{R}\}$.

Remarks:

- Γ is parameterized by s for clarity, this parameter will not be used for any other purpose.
- The solution u must satisfy the condition

  $$u(x, \sin x) = 0$$

  for all x.

Solution (1 of 5)

The characteristic system for this example is

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x, \quad \frac{du}{dt} = e^u.$$

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From the second equation,

$$\frac{dy}{dt} = -x = \frac{d}{dt}\left[\frac{dx}{dt}\right] = \frac{d^2x}{dt^2} \iff x''(t) + x = 0.$$

Thus $x(t) = A\cos t + B\sin t$ and consequently
$y(t) = x'(t) = -A\sin t + B\cos t$ where A and B are constants.

Solution (2 of 5)

The characteristic curves of the solution can be parameterized as

$$\begin{aligned} x &= A\cos t + B\sin t \\ y &= B\cos t - A\sin t \\ u &= -\ln(C - t). \end{aligned}$$

Solution (2 of 5)

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u &= -\ln(C - t).
\end{aligned}
$$

The non-characteristic curve Γ is parameterized as

$$
\begin{aligned}
x &= s \\
y &= \sin s \\
u &= 0.
\end{aligned}
$$

For each point on Γ we want to find a characteristic curve which passes through the point when $t = 0$ (arbitrary choice).

Solution (3 of 5)

When $t = 0$,

$$
\begin{aligned}
x(0) &= s = A \\
y(0) &= \sin s = B \\
u(0) &= 0 = -\ln C \iff C = 1
\end{aligned}
$$

The characteristic curve intersecting Γ at $t = 0$ can be parameterized as

$$
\begin{aligned}
x(t) &= s \cos t + \sin s \sin t \\
y(t) &= \sin s \cos t - s \sin t \\
u(t) &= -\ln(1 - t).
\end{aligned}
$$

These equations also produce the integral surface for the solution of the quasilinear PDE.

Solution (4 of 5)

$$\begin{aligned} x &= s\cos t + \sin s \sin t \\ y &= \sin s \cos t - s \sin t \\ u &= -\ln(1-t). \end{aligned}$$

We can eliminate s and t from the equations. Multiply the first equation by $\cos t$ and the second equation by $-\sin t$ and add them together.

$$\begin{aligned} x\cos t - y \sin t &= s\cos^2 t + \sin s \cos t \sin t - \sin s \cos t \sin t + s \sin^2 t \\ &= s \end{aligned}$$

Multiply the first equation by $\sin t$ and the second equation by $\cos t$ and add them together.

$$\begin{aligned} x \sin t + y \cos t &= s \cos t \sin t + \sin s \sin^2 t + \sin s \cos^2 - s \cos t \sin t \\ &= \sin s \end{aligned}$$

Combining the results produces:

$$\sin(x \cos t - y \sin t) = x \sin t + y \cos t.$$

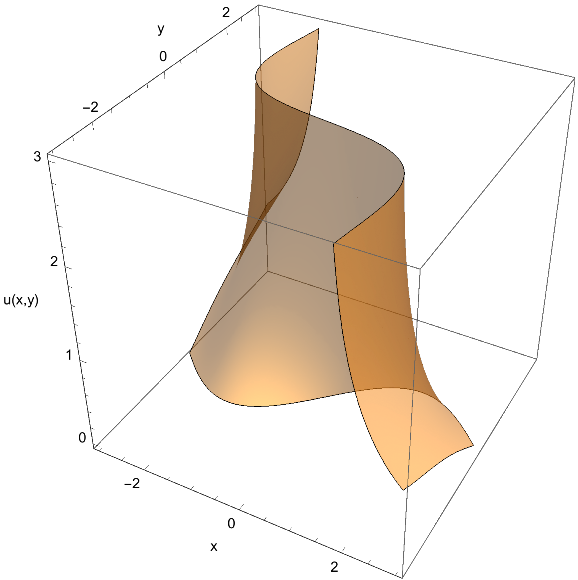
Solution (5 of 5)

We managed to eliminate parameter s. Using $u = -\ln(1-t) \iff t = 1 - e^{-u}$ we can eliminate parameter t as well.

The solution to the quasilinear PDE can be expressed in implicit form as

$$\begin{aligned} &\sin\left(x\cos\left[1-e^{-u}\right] - y\sin\left[1-e^{-u}\right]\right) \\ &= x\sin\left[1-e^{-u}\right] + y\cos\left[1-e^{-u}\right]. \end{aligned}$$

Illustration



Example

Consider the quasilinear first-order PDE

$$x\,u_x + y\,u_y - \sec u = 0.$$

Find the solution passing through the parametric curve Γ given by $\Gamma = \{(x, y, u) = (s^2, \sin s, 0)\ :\ s \in \mathbb{R}\}$.

Solution (1 of 3)

The characteristic system is:

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = y$$
$$\frac{du}{dt} = \sec u$$

Solution (1 of 3)

The characteristic system is:

$$\begin{aligned}
\frac{dx}{dt} &= x \implies x(t) = A\,e^{t} \\
\frac{dy}{dt} &= y \implies y(t) = B\,e^{t} \\
\frac{du}{dt} &= \sec u
\end{aligned}$$

Solution (1 of 3)

The characteristic system is:

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\frac{du}{dt} &= \sec u \implies \sin u = t + C
\end{aligned}$$

where A, B, and C are constants.

Solution (1 of 3)

The characteristic system is:

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where A, B, and C are constants.

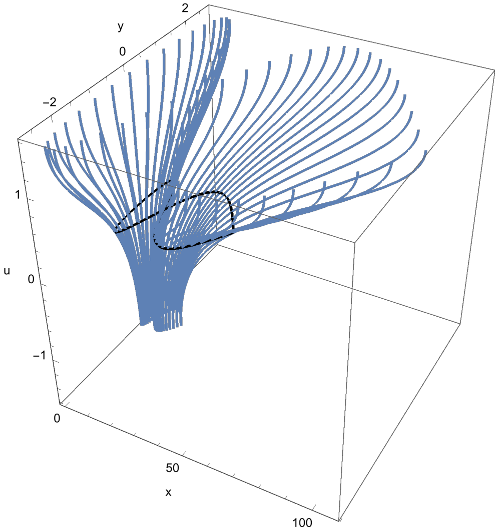
Suppose the characteristics pass through Γ at $t = 0$ (arbitrary choice).

$$\begin{aligned} x(0) &= A = s^2 \\ y(0) &= B = \sin s \\ u(0) &= 0 = 0 + C \implies C = 0 \end{aligned}$$

Solution (2 of 3)

The characteristic curves take the form:

$$x = s^2 e^t, \quad y = e^t \sin s, \quad u = \arcsin t.$$



Solution (3 of 3)

We can eliminate parameters s and t from the characteristics curves. Since $t = \sin u$ then

$$y = e^{\sin u} \sin s \iff \sin s = y\, e^{-\sin u} \implies s = \arcsin\left(y\, e^{-\sin u}\right).$$

Consequently the implicit form of the integral surface can be expressed as

$$x = \left[\arcsin\left(y\, e^{-\sin u}\right)\right]^2 e^{\sin u}.$$

Illustration

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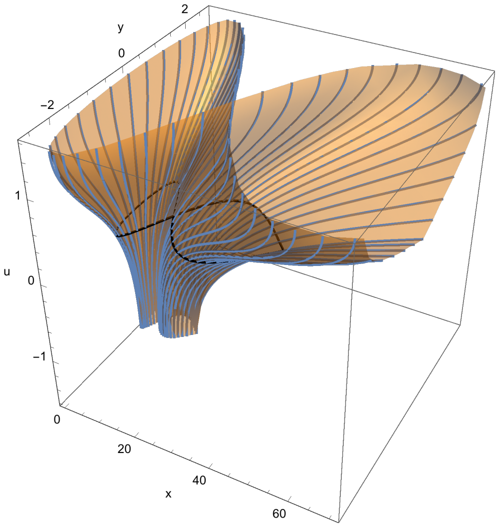

Example

Find the solution to the first-order, quasilinear PDE with side condition.

$$
\begin{aligned}
u_x + u\,u_y &= 6x \\
u(0, y) &= 3y
\end{aligned}
$$

Example

Find the solution to the first-order, quasilinear PDE with side condition.

$$\begin{aligned} u_x + u\,u_y &= 6x \\ u(0, y) &= 3y \end{aligned}$$

Remark: since no non-characteristic curve Γ is mentioned we are free to create one. A natural one to use would be $\Gamma = \{(x, y, u) = (0, s, 3s) \;:\; s \in \mathbb{R}\}$.

Solution (1 of 3)

Re-write the PDE as

$$u_x + u\,u_y - 6x = 0$$

which has the characteristic system:

$$\begin{aligned}\frac{dx}{dt} &= 1\\ \frac{dy}{dt} &= u\\ \frac{du}{dt} &= 6x\end{aligned} \qquad .$$

Solution (1 of 3)

Re-write the PDE as

$$u_x + u\,u_y - 6x = 0$$

which has the characteristic system:

$$\begin{aligned}\frac{dx}{dt} &= 1 \implies x = t + A\\ \frac{dy}{dt} &= u\\ \frac{du}{dt} &= 6x\end{aligned} \qquad .$$

Solution (1 of 3)

Re-write the PDE as

$$u_x + u\,u_y - 6x = 0$$

which has the characteristic system:

$$\begin{aligned}
\frac{dx}{dt} &= 1 \implies x = t + A \\
\frac{dy}{dt} &= u \\
\frac{du}{dt} &= 6x = 6t + 6A \implies u = 3t^2 + 6A\,t + B.
\end{aligned}$$

Solution (1 of 3)

Re-write the PDE as

$$u_x + u\,u_y - 6x = 0$$

which has the characteristic system:

$$\begin{aligned}
\frac{dx}{dt} &= 1 \implies x = t + A \\
\frac{dy}{dt} &= u = 3t^2 + 6A\,t + B \implies y = t^3 + 3A\,t^2 + B\,t + C \\
\frac{du}{dt} &= 6x = 6t + 6A \implies u = 3t^2 + 6A\,t + B.
\end{aligned}$$

Solution (2 of 3)

For each point on Γ find the characteristic curve which passes through the point when $t = 0$ (arbitrary choice).

$$\begin{aligned} x(0) &= 0 = A \\ y(0) &= s = C \\ u(0) &= 3s = B \end{aligned}$$

Consequently the characteristic curves are in parametric form:

$$\begin{aligned} x(t) &= t \\ y(t) &= t^3 + s(3t + 1) \\ u(t) &= 3t^2 + 3s. \end{aligned}$$

Solution (3 of 3)

Now eliminate s and t so as to write the solution in implicit form. Taking the parametric equations for x and u we can solve for s and t to obtain

$$\begin{aligned} s &= \frac{u}{3} - x^2 \\ t &= x. \end{aligned}$$

Substituting into the equation for y we get

$$y = \frac{u}{3} + x\,u - x^2 - 3x^3$$

which can be solved explicitly for

$$u(x,y) = \frac{3(2x^3 + x^2 + y)}{3x + 1}.$$

Illustration