Complex Form of Fourier Series

Partial Differential Equations

J Robert Buchanan

Department of Mathematics

Fall 2025

Objectives

In this lesson we will learn:

- the Euler identity,
- the complex function representation of Fourier series, and
- review the hyperbolic trigonometric functions.

Euler Identity

Let $i = \sqrt{-1}$, then for any real number θ,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Remark: this provides a connection between the exponential function and the ordinary trigonometric functions.

Related Identities

$$
\begin{aligned}
\cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
\sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}.
\end{aligned}
$$

Related Identities

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Note how these identities mirror the definitions of the hyperbolic trigonometric functions.

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$
$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}.$$

Transition to Complex Form (1 of 2)

Suppose $f(x)$ is integrable on $[-L, L]$.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

Transition to Complex Form (1 of 2)

Suppose $f(x)$ is integrable on $[-L, L]$.

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\begin{aligned}
f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} \left(e^{\frac{in\pi x}{L}} + e^{\frac{-in\pi x}{L}} \right) + \frac{b_n}{2i} \left(e^{\frac{in\pi x}{L}} - e^{\frac{-in\pi x}{L}} \right) \right]
\end{aligned}
$$

Transition to Complex Form (1 of 2)

Suppose $f(x)$ is integrable on $[-L, L]$.

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Transition to Complex Form (2 of 2)

Suppose $f(x)$ is integrable on $[-L, L]$.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n - ib_n}{2} e^{\frac{in\pi x}{L}} + \frac{a_n + ib_n}{2} e^{\frac{-in\pi x}{L}} \right]$$

Define $b_0 = 0$, $c_0 = \frac{a_0 + ib_0}{2}$, $c_n = \frac{a_n - ib_n}{2}$, and $c_{-n} = \frac{a_0 + ib_n}{2}$, then

$$f(x) \sim c_0 + \sum_{n=1}^{\infty} \left[c_n e^{\frac{in\pi x}{L}} + c_{-n} e^{\frac{-in\pi x}{L}} \right]$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}.$$

This is the Fourier series for $f(x)$ in complex form.

Complex Coefficients (1 of 2)

For $n = 0, 1, 2, \ldots$,

$$
\begin{aligned}
c_n = \frac{a_n - ib_n}{2} &= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx - \frac{i}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right) \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\left(\cos\frac{n\pi x}{L} - i\sin\frac{n\pi x}{L}\right)\,dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx.
\end{aligned}
$$

Complex Coefficients (2 of 2)

For $n = 1, 2, \ldots,$

$$
\begin{aligned}
c_{-n} = \frac{a_n + ib_n}{2} &= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx + \frac{i}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right) \\
&= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{-n\pi x}{L}\,dx - \frac{i}{L}\int_{-L}^{L} f(x)\sin\frac{-n\pi x}{L}\,dx\right) \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx.
\end{aligned}
$$

Complex Coefficients (2 of 2)

For $n = 1, 2, \ldots,$

$$c_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx + \frac{i}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right)$$

$$= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{-n\pi x}{L}\,dx - \frac{i}{L}\int_{-L}^{L} f(x)\sin\frac{-n\pi x}{L}\,dx\right)$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx.$$

Therefore for any $n \in \mathbb{Z}$,

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx.$$

Example

Suppose $i = \sqrt{-1}$ and a is a constant with $a \notin \{k\, i \mid k = 0, 1, 2, \ldots\}$. Find the complex form of the Fourier series of $f(x) = e^{ax}$ on $(-\pi, \pi)$.

Solution

Using the formula for the complex Fourier coefficients,

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-inx} \, dx \\
&= \frac{1}{2(a-in)\pi} \left(e^{(a-in)\pi} - e^{-(a-in)\pi} \right) \\
&= \frac{1}{(a-in)\pi} \left(\frac{e^{(a-in)\pi} - e^{-(a-in)\pi}}{2} \right) \\
&= \frac{\sinh(a-in)\pi}{(a-in)\pi}.
\end{aligned}$$

Solution

Using the formula for the complex Fourier coefficients,

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\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}\,dx \\
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&= \frac{1}{(a-in)\pi}\left(\frac{e^{(a-in)\pi} - e^{-(a-in)\pi}}{2}\right) \\
&= \frac{\sinh(a-in)\pi}{(a-in)\pi}.
\end{aligned}
$$

Therefore the complex form of the Fourier series is

$$
e^{ax} \sim \sum_{n=-\infty}^{\infty} \frac{\sinh(a-in)\pi}{(a-in)\pi} e^{inx}.
$$