

<div data-bbox="194 1276 288 2016"> <h1>Differentiation and Integration of Fourier Series</h1> <h2>MATH 467 Partial Differential Equations</h2> </div> <div data-bbox="408 1520 576 1771"> <p>J Robert Buchanan</p> <p>Department of Mathematics</p> <p>Fall 2022</p> </div>	<div data-bbox="25 902 65 1095"> <h3>Objectives</h3> </div> <div data-bbox="288 322 448 1037"> <p>In this lesson we will learn about</p> <ul style="list-style-type: none"> ▶ the properties of the derivatives of Fourier series, ▶ the properties of the integrals of Fourier series, and ▶ Parseval's Identity and Bessel's Inequality. </div>
<div data-bbox="821 1774 865 2150"> <h3>Cautionary Example</h3> </div> <div data-bbox="919 1245 1508 2074"> <ol style="list-style-type: none"> Find the Fourier (sine) series for the 2π-periodic extension of $f(x) = x$. <div data-bbox="1018 1245 1189 1995"> $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx = \frac{2(-1)^{n+1}}{n}$ $f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$ </div> Differentiate the series term by term. <div data-bbox="1305 1489 1390 1751"> $2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos(nx)$ </div> Does the result converge to $f'(x) = 1$? No, in fact the result diverges for all $x \in (-\pi, \pi)$. </div>	<div data-bbox="821 519 861 1095"> <h3>Differentiation of Fourier Series</h3> </div> <div data-bbox="1101 156 1241 1037"> <p>Theorem Assume that $f(x)$ is continuous on $(-L, L)$ with $f(-L+) = f(L-)$, and $f'(x)$ is piecewise continuous on $(-L, L)$. Then the Fourier series of $f(x)$ can be differentiated term-by-term.</p> </div>

Example

- ▶ The Fourier (cosine) series for the 2π -periodic extension of $F(x) = x^2$ is

$$F(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

- ▶ The Fourier (sine) series for the 2π -periodic extension of $f(x) = 2x$ is

$$f(x) = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

- ▶ **Note:** the term-by-term derivative of the Fourier series for $F(x)$ results in the Fourier series for $f(x) = F'(x)$.

Proof (1 of 2)

- ▶ Suppose $f(x)$ is continuous on $(-L, L)$ and $f(-L+) = f(L-)$.
- ▶ Define $f(L) = f(L-) = f(-L+)$ which will make the $2L$ -periodic extension of $f(x)$ continuous on $(-\infty, \infty)$.
- ▶ The Fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

- ▶ If $f'(x)$ is piecewise continuous then $f'(x)$ has a Fourier series representation and

$$f'(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\alpha_n \cos \frac{n\pi x}{L} + \beta_n \sin \frac{n\pi x}{L} \right).$$

- except at the removable or jump discontinuities of $f'(x)$.
- ▶ We need to show $\alpha_0 = 0$ and

$$\alpha_n = \frac{n\pi}{L} b_n \quad \text{and} \quad \beta_n = -\frac{n\pi}{L} a_n.$$

Proof (2 of 2)

$$\begin{aligned} \alpha_0 &= \frac{1}{L} \int_{-L}^L f'(x) dx = \frac{1}{L} [f(x)]_{-L}^L = 0 \\ \alpha_n &= \frac{1}{L} \int_{-L}^L f'(x) \cos \frac{n\pi x}{L} dx \\ &= \frac{1}{L} \left[f(x) \cos \frac{n\pi x}{L} \right]_{-L}^L + \frac{n\pi}{L} \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \\ &= \frac{n\pi}{L} b_n \end{aligned}$$

Using integration by parts. Similarly it is shown that $\beta_n = -\frac{n\pi}{L} a_n$.

Consequence of Differentiation Theorem

The assumptions of the theorem include the continuity of the $2L$ -periodic extension of $f(x)$ and the piecewise continuity of the $2L$ -periodic extension of $f'(x)$. Thus the Dirichlet Convergence Theorem states

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

for all $x \in (-\infty, \infty)$. The Fourier series for $f'(x)$ is

$$f'(x) \sim \frac{\pi}{L} \sum_{n=1}^{\infty} \left(n b_n \cos \frac{n\pi x}{L} - n a_n \sin \frac{n\pi x}{L} \right),$$

and for any $(-\infty, \infty)$

$$\frac{f'(x+) + f'(x-)}{2} = \frac{\pi}{L} \sum_{n=1}^{\infty} \left(n b_n \cos \frac{n\pi x}{L} - n a_n \sin \frac{n\pi x}{L} \right)$$

Integration of Fourier Series

Theorem

Let $f(x)$ be piecewise continuous on $[-L, L]$. The Fourier series of $f(x)$ can be integrated term by term and the resulting series always converges to the integral of $f(x)$ on $[-L, L]$. That is, if

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

then

$$\int_0^x f(s) ds = \frac{a_0}{2}x + \frac{L}{\pi} \sum_{n=1}^{\infty} \left[\frac{a_n}{n} \sin \frac{n\pi x}{L} + \frac{b_n}{n} \left(1 - \cos \frac{n\pi x}{L} \right) \right].$$

Proof (1 of 4)

- ▶ Define $F(x) = \int_0^x f(s) ds$ and define $G(x) = F(x) - a_0x/2$.
- ▶ Since $f'(x)$ is piecewise continuous on $[-L, L]$ then $F(x)$ and $G(x)$ are continuous on $(-L, L)$.
- ▶ By definition

$$a_0 = \frac{1}{L} \int_{-L}^L f(s) ds$$

$$\frac{a_0L}{2} + \frac{a_0L}{2} = - \int_0^{-L} f(s) ds + \int_0^L f(s) ds$$

$$\int_0^{-L} f(s) ds + \frac{a_0L}{2} = G(-L) = G(L) = \int_0^L f(s) ds - \frac{a_0L}{2}.$$

- ▶ Therefore $G(x)$ has a Fourier series

$$G(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right).$$

Proof (2 of 4)

The Fourier coefficients for $G(x)$ are by definition,

$$\begin{aligned} A_n &= \frac{1}{L} \int_{-L}^L G(x) \cos \frac{n\pi x}{L} dx \\ &= \frac{1}{n\pi} \left(\left[\left(G(x) \sin \frac{n\pi x}{L} \right) \right]_{-L}^L - \int_{-L}^L \left(f(x) - \frac{a_0}{2} \right) \sin \frac{n\pi x}{L} dx \right) \\ &= -\frac{1}{n\pi} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx + \frac{a_0}{2n\pi} \int_{-L}^L \sin \frac{n\pi x}{L} dx \\ &= -\frac{L}{n\pi} b_n \end{aligned}$$

using integration by parts. Similarly it can be shown that $B_n = \frac{L}{n\pi} a_n$.

Proof (3 of 4)

Thus far we have shown

$$\begin{aligned} G(x) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(-\frac{L}{n\pi} b_n \cos \frac{n\pi x}{L} + \frac{L}{n\pi} a_n \sin \frac{n\pi x}{L} \right) \\ G(0) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(-\frac{L}{n\pi} b_n \cos \frac{n\pi(0)}{L} + \frac{L}{n\pi} a_n \sin \frac{n\pi(0)}{L} \right) \\ 0 &= \frac{A_0}{2} - \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n \\ \frac{A_0}{2} &= \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n. \end{aligned}$$

Consequently

$$\begin{aligned} G(x) &= \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n + \sum_{n=1}^{\infty} \left(-\frac{L}{n\pi} b_n \cos \frac{n\pi x}{L} + \frac{L}{n\pi} a_n \sin \frac{n\pi x}{L} \right) \\ &= \sum_{n=1}^{\infty} \frac{L}{n\pi} \left(b_n \left(1 - \cos \frac{n\pi x}{L} \right) + \frac{L}{n\pi} a_n \sin \frac{n\pi x}{L} \right) \end{aligned}$$

$$F(x) = \frac{a_0}{2} x + \frac{L}{\pi} \sum_{n=1}^{\infty} \left(\frac{b_n}{n} \left(1 - \cos \frac{n\pi x}{L} \right) + \frac{a_n}{n} \sin \frac{n\pi x}{L} \right).$$

Let the N th partial sum of the Fourier series for $f(x)$ be

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^N \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

- ▶ We wish to describe the error present in this approximation.
- ▶ Multiply S_N by $f(x)$ and integrate over $[-L, L]$.
- ▶ Multiply S_N by $S_N(x)$ and integrate over $[-L, L]$.

Multiplication by $f(x)$

$$\begin{aligned} &\int_{-L}^L f(x) S_N(x) dx \\ &= \frac{1}{2} a_0 \int_{-L}^L f(x) dx \\ &\quad + \sum_{n=1}^N \left(a_n \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx + b_n \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \right) \\ &= \frac{L}{2} a_0^2 + L \sum_{n=1}^N (a_n^2 + b_n^2). \end{aligned}$$

Multiplication by $S_N(x)$

$$\begin{aligned} \int_{-L}^L S_N(x) S_N(x) dx &= \frac{1}{2} a_0 \int_{-L}^L S_N(x) dx \\ &\quad + \sum_{n=1}^N \left(a_n \int_{-L}^L S_N(x) \cos \frac{n\pi x}{L} dx + b_n \int_{-L}^L S_N(x) \sin \frac{n\pi x}{L} dx \right) \\ &= \frac{L}{2} a_0^2 + L \sum_{n=1}^N (a_n^2 + b_n^2). \end{aligned}$$

This is the same result as before.

Squared Error

One measure of the error in approximating $f(x)$ by $S_N(x)$ is

$$\begin{aligned}
 0 &\leq \int_{-L}^L (f(x) - S_N(x))^2 dx \\
 &= \int_{-L}^L (f(x))^2 dx - 2 \int_{-L}^L f(x) S_N(x) dx \\
 &\quad + \int_{-L}^L (S_N(x))^2 dx \\
 &= \int_{-L}^L (f(x))^2 dx - \left(\frac{L}{2} a_0^2 + L \sum_{n=1}^N (a_n^2 + b_n^2) \right) \\
 \frac{1}{2} a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) &\leq \frac{1}{L} \int_{-L}^L (f(x))^2 dx.
 \end{aligned}$$

This is known as **Bessel's inequality**.

Remarks

$$\begin{aligned}
 \sum_{n=1}^N (a_n^2 + b_n^2) &\leq \frac{1}{L} \int_{-L}^L (f(x))^2 dx \\
 \lim_{N \rightarrow \infty} \sum_{n=1}^N (a_n^2 + b_n^2) &\leq \frac{1}{L} \int_{-L}^L (f(x))^2 dx \\
 \sum_{n=1}^{\infty} (a_n^2 + b_n^2) &\leq \frac{1}{L} \int_{-L}^L (f(x))^2 dx
 \end{aligned}$$

- ▶ A function for which $\int_{-L}^L (f(x))^2 dx$ is finite is said to be **square integrable** on $[-L, L]$.
- ▶ If $f(x)$ is square integrable on $[-L, L]$ then the infinite series $\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$ converges.
- ▶ The convergence of the infinite series implies $\lim_{n \rightarrow \infty} a_n = 0 = \lim_{n \rightarrow \infty} b_n$.

Mean Square Error

For $N \in \mathbb{N}$ define the **mean square error** of the partial sum $S_N(x)$ relative to $f(x)$ to be

$$E_N = \frac{1}{2L} \int_{-L}^L (f(x) - S_N(x))^2 dx.$$

Theorem

Assume that $f(x)$ is a square integrable function on $[-L, L]$, then

$$\lim_{N \rightarrow \infty} E_N = 0.$$

Corollary

Corollary

If $f(x)$ is a square integrable function on $[-L, L]$, then

$$\frac{1}{2L} \int_{-L}^L (f(x))^2 dx = \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

where a_n and b_n are the Fourier coefficients of $f(x)$.

Remark: this equation is known as **Parseval's identity** and functions like the Pythagorean theorem for Fourier series.

Application of Parseval's Identity

Find the $\sum_{n=1}^{\infty} \frac{1}{n^6}$ using

- ▶ the Fourier (sine) coefficients for $f(x) = x^3$ on $(-\pi, \pi)$,
- ▶ Parseval's identity,
- ▶ and the facts that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Proof of Parseval's Identity

By definition

$$\begin{aligned}
 E_N &= \frac{1}{2L} \int_{-L}^L (f(x) - S_N(x))^2 dx \\
 &= \frac{1}{2L} \int_{-L}^L (f(x))^2 dx - \frac{1}{L} \int_{-L}^L f(x) S_N(x) dx + \frac{1}{2L} \int_{-L}^L (S_N(x))^2 dx \\
 &= \frac{1}{2L} \int_{-L}^L (f(x))^2 dx - \left[\frac{1}{2} a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) \right] + \left[\frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^N (a_n^2 + b_n^2) \right] \\
 &= \frac{1}{2L} \int_{-L}^L (f(x))^2 dx - \frac{1}{4} a_0^2 - \frac{1}{2} \sum_{n=1}^N (a_n^2 + b_n^2).
 \end{aligned}$$

Since $\lim_{N \rightarrow \infty} E_N = 0$ then

$$\frac{1}{2L} \int_{-L}^L (f(x))^2 dx = \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Homework

- ▶ Read Sections 3.8 and 3.9
- ▶ Exercises: 20–25