

# Differentiation and Integration of Fourier Series

*Partial Differential Equations*

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# Objectives

In this lesson we will learn about

- ▶ the properties of the derivatives of Fourier series,
- ▶ the properties of the integrals of Fourier series, and
- ▶ Parseval's Identity and Bessel's Inequality.

# Cautionary Example

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2. Differentiate the series term by term.

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3. Does the result converge to  $f'(x) = 1$ ? No, in fact the result diverges for all  $x \in (-\pi, \pi)$ .

# Differentiation of Fourier Series

## Theorem

*Assume that  $f(x)$  is continuous on  $(-L, L)$  with  $f(-L+) = f(L-)$ , and  $f'(x)$  is piecewise continuous on  $(-L, L)$ . Then the Fourier series of  $f(x)$  can be differentiated term-by-term.*

# Example

- ▶ The Fourier (cosine) series for the  $2\pi$ -periodic extension of  $F(x) = x^2$  is

$$F(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

- ▶ The Fourier (sine) series for the  $2\pi$ -periodic extension of  $f(x) = 2x$  is

$$f(x) = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

- ▶ **Note:** the term-by-term derivative of the Fourier series for  $F(x)$  results in the Fourier series for  $f(x) = F'(x)$ .

## Proof (1 of 2)

- ▶ Suppose  $f(x)$  is continuous on  $(-L, L)$  and  $f(-L+) = f(L-)$ .
- ▶ Define  $f(L) = f(L-) = f(-L+)$  which will make the  $2L$ -periodic extension of  $f(x)$  continuous on  $(-\infty, \infty)$ .
- ▶ The Fourier series for  $f(x)$  is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

- ▶ If  $f'(x)$  is piecewise continuous then  $f'(x)$  has a Fourier series representation and

$$f'(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left( \alpha_n \cos \frac{n\pi x}{L} + \beta_n \sin \frac{n\pi x}{L} \right).$$

except at the removable or jump discontinuities of  $f'(x)$ .

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- ▶ We need to show  $\alpha_0 = 0$  and

$$\alpha_n = \frac{n\pi}{L} b_n \text{ and } \beta_n = -\frac{n\pi}{L} a_n.$$

## Proof (2 of 2)

$$\alpha_0 = \frac{1}{L} \int_{-L}^L f'(x) dx = \frac{1}{L} [f(x)]_{x=-L}^{x=L} = 0$$

$$\begin{aligned}\alpha_n &= \frac{1}{L} \int_{-L}^L f'(x) \cos \frac{n\pi x}{L} dx \\&= \frac{1}{L} \left[ f(x) \cos \frac{n\pi x}{L} \right]_{x=-L}^{x=L} + \frac{n\pi}{L} \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \\&= \frac{n\pi}{L} b_n\end{aligned}$$

Using integration by parts. Similarly it is shown that  $\beta_n = -\frac{n\pi}{L} a_n$ .

# Consequence of Differentiation Theorem

The assumptions of the theorem include the continuity of the  $2L$ -periodic extension of  $f(x)$  and the piecewise continuity of the  $2L$ -periodic extension of  $f'(x)$ . Thus the Dirichlet Convergence Theorem states

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

for all  $x \in (-\infty, \infty)$ . The Fourier series for  $f'(x)$  is

$$f'(x) \sim \frac{\pi}{L} \sum_{n=1}^{\infty} \left( n b_n \cos \frac{n\pi x}{L} - n a_n \sin \frac{n\pi x}{L} \right),$$

and for any  $(-\infty, \infty)$

$$\frac{f'(x+) + f'(x-)}{2} = \frac{\pi}{L} \sum_{n=1}^{\infty} \left( n b_n \cos \frac{n\pi x}{L} - n a_n \sin \frac{n\pi x}{L} \right)$$

# Integration of Fourier Series

## Theorem

*Let  $f(x)$  be piecewise continuous on  $[-L, L]$ . The Fourier series of  $f(x)$  can be integrated term by term and the resulting series always converges to the integral of  $f(x)$  on  $[-L, L]$ . That is, if*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

*then*

$$\int_0^x f(s) ds = \frac{a_0}{2}x + \frac{L}{\pi} \sum_{n=1}^{\infty} \left[ \frac{a_n}{n} \sin \frac{n\pi x}{L} + \frac{b_n}{n} \left( 1 - \cos \frac{n\pi x}{L} \right) \right].$$

## Proof (1 of 4)

- ▶ Define  $F(x) = \int_0^x f(s) ds$  and define  $G(x) = F(x) - a_0x/2$ .
- ▶ Since  $f(x)$  is piecewise continuous on  $[-L, L]$  then  $F(x)$  and  $G(x)$  are continuous on  $(-L, L)$ .
- ▶ By definition

$$a_0 = \frac{1}{L} \int_{-L}^L f(s) ds$$

$$\frac{a_0L}{2} + \frac{a_0L}{2} = - \int_0^{-L} f(s) ds + \int_0^L f(s) ds$$

$$\int_0^{-L} f(s) ds + \frac{a_0L}{2} = G(-L) = G(L) = \int_0^L f(s) ds - \frac{a_0L}{2}.$$

- ▶ Therefore  $G(x)$  has a Fourier series

$$G(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right).$$

## Proof (2 of 4)

The Fourier coefficients for  $G(x)$  are by definition,

$$\begin{aligned}A_n &= \frac{1}{L} \int_{-L}^L G(x) \cos \frac{n\pi x}{L} dx \\&= \frac{1}{n\pi} \left( \left[ \left( G(x) \sin \frac{n\pi x}{L} \right) \right]_{x=-L}^{x=L} - \int_{-L}^L \left( f(x) - \frac{a_0}{2} \right) \sin \frac{n\pi x}{L} dx \right) \\&= -\frac{1}{n\pi} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx + \frac{a_0}{2n\pi} \int_{-L}^L \sin \frac{n\pi x}{L} dx \\&= -\frac{L}{n\pi} b_n\end{aligned}$$

using integration by parts. Similarly it can be shown that  $B_n = \frac{L}{n\pi} a_n$ .

## Proof (3 of 4)

Thus far we have shown

$$G(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( -\frac{L}{n\pi} b_n \cos \frac{n\pi x}{L} + \frac{L}{n\pi} a_n \sin \frac{n\pi x}{L} \right)$$

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$$G(0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( -\frac{L}{n\pi} b_n \cos \frac{n\pi(0)}{L} + \frac{L}{n\pi} a_n \sin \frac{n\pi(0)}{L} \right)$$

$$0 = \frac{A_0}{2} - \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n$$

$$\frac{A_0}{2} = \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n.$$

# Proof (4 of 4)

Consequently

$$\begin{aligned} G(x) &= \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n + \sum_{n=1}^{\infty} \left( -\frac{L}{n\pi} b_n \cos \frac{n\pi x}{L} + \frac{L}{n\pi} a_n \sin \frac{n\pi x}{L} \right) \\ &= \sum_{n=1}^{\infty} \frac{L}{n\pi} \left( b_n \left( 1 - \cos \frac{n\pi x}{L} \right) + \frac{L}{n\pi} a_n \sin \frac{n\pi x}{L} \right) \\ F(x) &= \frac{a_0}{2} x + \frac{L}{\pi} \sum_{n=1}^{\infty} \left( \frac{b_n}{n} \left( 1 - \cos \frac{n\pi x}{L} \right) + \frac{a_n}{n} \sin \frac{n\pi x}{L} \right). \end{aligned}$$

# Partial Sum of Fourier Series

Let the  $N$ th partial sum of the Fourier series for  $f(x)$  be

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^N \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

- ▶ We wish to describe the error present in this approximation.
- ▶ Multiply  $S_N$  by  $f(x)$  and integrate over  $[-L, L]$ .
- ▶ Multiply  $S_N$  by  $S_N(x)$  and integrate over  $[-L, L]$ .

## Multiplication by $f(x)$

$$\begin{aligned}& \int_{-L}^L f(x) S_N(x) dx \\&= \frac{1}{2} a_0 \int_{-L}^L f(x) dx \\&\quad + \sum_{n=1}^N \left( a_n \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx + b_n \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \right) \\&= \frac{L}{2} a_0^2 + L \sum_{n=1}^N (a_n^2 + b_n^2) .\end{aligned}$$

## Multiplication by $S_N(x)$

$$\begin{aligned}\int_{-L}^L S_N(x) S_N(x) dx &= \frac{1}{2} a_0 \int_{-L}^L S_N(x) dx \\ &\quad + \sum_{n=1}^N \left( a_n \int_{-L}^L S_N(x) \cos \frac{n\pi x}{L} dx + b_n \int_{-L}^L S_N(x) \sin \frac{n\pi x}{L} dx \right) \\ &= \frac{L}{2} a_0^2 + L \sum_{n=1}^N (a_n^2 + b_n^2).\end{aligned}$$

This is the same result as before.

# Squared Error

One measure of the error in approximating  $f(x)$  by  $S_N(x)$  is

$$0 \leq \int_{-L}^L (f(x) - S_N(x))^2 dx$$

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$$\begin{aligned} 0 &\leq \int_{-L}^L (f(x) - S_N(x))^2 dx \\ &= \int_{-L}^L (f(x))^2 dx - 2 \int_{-L}^L f(x) S_N(x) dx \\ &\quad + \int_{-L}^L (S_N(x))^2 dx \\ &= \int_{-L}^L (f(x))^2 dx - \left( \frac{L}{2} a_0^2 + L \sum_{n=1}^N (a_n^2 + b_n^2) \right) \end{aligned}$$

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$$\frac{1}{2} a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) \leq \frac{1}{L} \int_{-L}^L (f(x))^2 dx.$$

This is known as **Bessel's inequality**.

## Remarks

$$\sum_{n=1}^N (a_n^2 + b_n^2) \leq \frac{1}{L} \int_{-L}^L (f(x))^2 dx$$

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N (a_n^2 + b_n^2) \leq \frac{1}{L} \int_{-L}^L (f(x))^2 dx$$

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{L} \int_{-L}^L (f(x))^2 dx$$

- A function for which  $\int_{-L}^L (f(x))^2 dx$  is finite is said to be **square integrable on  $[-L, L]$** .

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- ▶ If  $f(x)$  is square integrable on  $[-L, L]$  then the infinite series  $\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$  converges.

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- ▶ If  $f(x)$  is square integrable on  $[-L, L]$  then the infinite series  $\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$  converges.
- ▶ The convergence of the infinite series implies  $\lim_{n \rightarrow \infty} a_n = 0 = \lim_{n \rightarrow \infty} b_n$ .

# Mean Square Error

For  $N \in \mathbb{N}$  define the **mean square error** of the partial sum  $S_N(x)$  relative to  $f(x)$  to be

$$E_N = \frac{1}{2L} \int_{-L}^L (f(x) - S_N(x))^2 dx.$$

## Theorem

*Assume that  $f(x)$  is a square integrable function on  $[-L, L]$ , then*

$$\lim_{N \rightarrow \infty} E_N = 0.$$

# Corollary

## Corollary

*If  $f(x)$  is a square integrable function on  $[-L, L]$ , then*

$$\frac{1}{2L} \int_{-L}^L (f(x))^2 dx = \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

*where  $a_n$  and  $b_n$  are the Fourier coefficients of  $f(x)$ .*

**Remark:** this equation is known as **Parseval's identity** and functions like the Pythagorean theorem for Fourier series.

# Application of Parseval's Identity

Find the  $\sum_{n=1}^{\infty} \frac{1}{n^6}$  using

- ▶ the Fourier (sine) coefficients for  $f(x) = x^3$  on  $(-\pi, \pi)$ ,
- ▶ Parseval's identity,
- ▶ and the facts that  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$  and  $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ .

# Solution

Fourier coefficients:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} x^3 \sin(nx) dx = \frac{2(-1)^n(6 - n^2\pi^2)}{n^3}$$

Parseval's identity:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^3)^2 dx &= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{2(-1)^n(6 - n^2\pi^2)}{n^3} \right]^2 \\ \frac{\pi^6}{7} &= \sum_{n=1}^{\infty} \frac{72}{n^6} - \sum_{n=1}^{\infty} \frac{24\pi^2}{n^4} + \sum_{n=1}^{\infty} \frac{2\pi^4}{n^2} \\ \frac{\pi^6}{7} + \frac{4\pi^6}{15} - \frac{\pi^6}{3} &= 72 \sum_{n=1}^{\infty} \frac{1}{n^6} \\ \frac{\pi^6}{945} &= \sum_{n=1}^{\infty} \frac{1}{n^6} \end{aligned}$$

# Proof of Parseval's Identity

By definition

$$\begin{aligned} E_N &= \frac{1}{2L} \int_{-L}^L (f(x) - S_N(x))^2 dx \\ &= \frac{1}{2L} \int_{-L}^L (f(x))^2 dx - \frac{1}{L} \int_{-L}^L f(x) S_N(x) dx + \frac{1}{2L} \int_{-L}^L (S_N(x))^2 dx \\ &= \frac{1}{2L} \int_{-L}^L (f(x))^2 dx - \left[ \frac{1}{2} a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) \right] + \left[ \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^N (a_n^2 + b_n^2) \right] \\ &= \frac{1}{2L} \int_{-L}^L (f(x))^2 dx - \frac{1}{4} a_0^2 - \frac{1}{2} \sum_{n=1}^N (a_n^2 + b_n^2). \end{aligned}$$

Since  $\lim_{N \rightarrow \infty} E_N = 0$  then

$$\frac{1}{2L} \int_{-L}^L (f(x))^2 dx = \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$