Dirichlet Convergence Theorem

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will sketch proofs of

- the Dirichlet Convergence Theorem for Fourier series, and
- the theorem regarding the uniform convergence of Fourier series.

Dirichlet Convergence Theorem

Theorem

Assume that $f(x)$ is a piecewise smooth function on the interval $[-L, L]$ extended to $(-\infty, \infty)$ periodically with period $2L$. Then the Fourier series of $f(x)$ converges for all x to the value

$$\frac{1}{2}\left(f(x+) + f(x-)\right).$$

First Necessary Lemma

In order to prove Dirichlet's convergence theorem for Fourier series we will need the following lemma.

Lemma

For any fixed $L > 0$ and piecewise continuous function $g(x)$ on interval $[0, L]$,

$$\lim_{N\to\infty} \int_0^L g(t) \sin \frac{(2N+1)\pi t}{2L} \, dt = 0.$$

Proof

Let $g(x)$ be piecewise continuous on $[0, L]$.

$$\int_0^L g(t)\sin\frac{(2N+1)\pi t}{L}\,dt$$
$$= \int_0^L \left[g(t)\sin\frac{\pi t}{L}\right]\cos\frac{2N\pi t}{L}\,dt + \int_0^L \left[g(t)\cos\frac{\pi t}{L}\right]\sin\frac{2N\pi t}{L}\,dt$$

Consider the even $2L$-periodic extension of $g(t)\sin(\pi t/L)$ and the $2N$th coefficient in the Fourier cosine series for this extension.

$$a_{2N} = \frac{2}{L}\int_0^L g(t)\sin\frac{\pi t}{L}\cos\frac{2N\pi t}{L}\,dt$$
$$\frac{L}{2}a_{2N} = \int_0^L g(t)\sin\frac{\pi t}{L}\cos\frac{2N\pi t}{L}\,dt$$
$$\lim_{N\to\infty}\frac{L}{2}a_{2N} = \lim_{N\to\infty}\int_0^L g(t)\sin\frac{\pi t}{L}\cos\frac{2N\pi t}{L}\,dt = 0$$

In a similar manner the other integral converges to 0 as $N \to \infty$.

Nth Partial Sum Revisited

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$
$$= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{N}\left[\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt\right]\cos\frac{n\pi x}{L}$$
$$+ \frac{1}{L}\sum_{n=1}^{N}\left[\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt\right]\sin\frac{n\pi x}{L}$$
$$= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$
$$+ \frac{1}{L}\sum_{n=1}^{N}\int_{-L}^{L} f(t)\left[\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}\right]dt$$
$$= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{N}\int_{-L}^{L} f(t)\cos\frac{n\pi(t-x)}{L}\,dt$$

Dirichlet Kernel

$$\begin{aligned}
S_N(x) &= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{N}\int_{-L}^{L} f(t)\cos\frac{n\pi(t-x)}{L}\,dt \\
&= \frac{1}{L}\int_{-L}^{L} f(t)\left[\frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi(t-x)}{L}\right]dt \\
&= \frac{1}{L}\int_{-L}^{L} f(t)D_N(t-x)\,dt
\end{aligned}$$

where

$$D_N(z) = \frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi z}{L}$$

is called the **Dirichlet Kernel**.

Closed Form of Dirichlet Kernel (1 of 3)

$$\begin{aligned}
D_N(z) &= \frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi z}{L} \\
D_N(z)\sin\frac{\pi z}{L} &= \frac{1}{2}\sin\frac{\pi z}{L} + \sum_{n=1}^{N}\cos\frac{n\pi z}{L}\sin\frac{\pi z}{L} \\
&= \frac{1}{2}\sin\frac{\pi z}{L} + \frac{1}{2}\sum_{n=1}^{N}\left[\sin\frac{(n+1)\pi z}{L} - \sin\frac{(n-1)\pi z}{L}\right]
\end{aligned}$$

according to a product-to-sum formula. The summation telescopes to produce

$$D_N(z)\sin\frac{\pi z}{L} = \frac{1}{2}\left[\sin\frac{(N+1)\pi z}{L} + \sin\frac{N\pi z}{L}\right].$$

Closed Form of Dirichlet Kernel (2 of 3)

$$
\begin{aligned}
&D_N(z)\sin\frac{\pi z}{L}\\
&= \frac{1}{2}\left[\sin\frac{(N+1)\pi z}{L} + \sin\frac{N\pi z}{L}\right]\\
&= \frac{1}{2}\left[\sin\frac{2(N+1)\pi z}{2L} + \sin\frac{2N\pi z}{2L}\right]\\
&= \frac{1}{2}\left[\sin\left(\frac{(2N+1)\pi z}{2L} + \frac{\pi z}{2L}\right) + \sin\left(\frac{(2N+1)\pi z}{2L} - \frac{\pi z}{2L}\right)\right]\\
&= \sin\frac{(2N+1)\pi z}{2L}\cos\frac{\pi z}{2L}
\end{aligned}
$$

using a product-to-sum formula.

Closed Form of Dirichlet Kernel (3 of 3)

$$
\begin{aligned}
D_N(z)\sin\frac{\pi z}{L} &= \sin\frac{(2N+1)\pi z}{2L}\cos\frac{\pi z}{2L}\\
D_N(z) &= \frac{\sin\frac{(2N+1)\pi z}{2L}\cos\frac{\pi z}{2L}}{\sin\frac{\pi z}{L}} \qquad (\text{if } z \neq k\,L \text{ with } k \in \mathbb{Z})\\
&= \frac{\sin\frac{(2N+1)\pi z}{2L}\cos\frac{\pi z}{2L}}{\sin\frac{2\pi z}{2L}}\\
&= \frac{\sin\frac{(2N+1)\pi z}{2L}\cos\frac{\pi z}{2L}}{2\sin\frac{\pi z}{2L}\cos\frac{\pi z}{2L}}\\
D_N(z) &= \frac{\sin\frac{(2N+1)\pi z}{2L}}{2\sin\frac{\pi z}{2L}}
\end{aligned}
$$

using the double-angle formula for the sine and assuming $z \neq 2kL$.

Remarks (1 of 2)

Consider the following limit:

$$\begin{aligned}
\lim_{z \to 2kL} D_N(z) &= \lim_{z \to 2kL} \frac{\sin \frac{(2N+1)\pi z}{2L}}{2 \sin \frac{\pi z}{2L}} \\
&= \lim_{z \to 2kL} \frac{\frac{(2N+1)\pi}{2L} \cos \frac{(2N+1)\pi z}{2L}}{\frac{\pi}{L} \cos \frac{\pi z}{2L}} \\
&= \lim_{z \to 2kL} \frac{(2N+1) \cos \frac{(2N+1)\pi z}{2L}}{2 \cos \frac{\pi z}{2L}} \\
&= \frac{2N+1}{2}
\end{aligned}$$

Thus the limit of the Dirichlet Kernel exists as $z \to 2kL$.

Remarks (2 of 2)

- It can also be shown that

  $$\lim_{z \to (2k+1)L} D_N(z) = \frac{(-1)^N}{2}$$

  and thus the closed form of the Dirichlet Kernel is defined for all real numbers.
- Integrating the original definition of the Dirichlet Kernel over $[0, L]$ yields

  $$\int_0^L D_N(z)\, dz = \frac{L}{2}.$$
- The Dirichlet Kernel is a $2L$-periodic and even function.

Second Necessary Lemma

Lemma

If $g(x)$ is a piecewise smooth function on $[0, L]$, then

$$\lim_{N\to\infty} \int_0^L g(t) D_N(t)\, dt = \frac{L}{2} g(0+).$$

Return to Proof of Dirichlet Convergence Theorem

Recall,

$$S_N(x) = \frac{1}{L} \int_{-L}^{L} f(t) D_N(t-x)\, dt.$$

- Since $f(t)D_N(t-x)$ is $2L$-periodic, then

$$S_N(x) = \frac{1}{L} \int_{x-L}^{x+L} f(t) D_N(t-x)\, dt.$$

- Make the substitution $s = t - x$,

$$\begin{aligned} S_N(x) &= \frac{1}{L} \int_{-L}^{L} f(x+s) D_N(s)\, ds \\ &= \frac{1}{L} \int_{-L}^{0} f(x+s) D_N(s)\, ds + \frac{1}{L} \int_0^L f(x+s) D_N(s)\, ds \\ &= \frac{1}{L} \int_0^L f(x-s) D_N(s)\, ds + \frac{1}{L} \int_0^L f(x+s) D_N(s)\, ds. \end{aligned}$$

Proof

Define,

$$I_N(x) = \frac{1}{L}\int_0^L f(x-s)D_N(s)\,ds$$
$$J_N(x) = \frac{1}{L}\int_0^L f(x+s)D_N(s)\,ds.$$

By the second necessary lemma,

$$\lim_{N\to\infty} I_N(x) = \lim_{N\to\infty} \frac{1}{L}\int_0^L f(x-s)D_N(s)\,ds = \frac{1}{L}\cdot\frac{L}{2}f(x-) = \frac{1}{2}f(x-)$$
$$\lim_{N\to\infty} J_N(x) = \lim_{N\to\infty} \frac{1}{L}\int_0^L f(x+s)D_N(s)\,ds = \frac{1}{L}\cdot\frac{L}{2}f(x+) = \frac{1}{2}f(x+),$$

and thus

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left[f(x-)+f(x+)\right].$$

Fourier Series Uniform Convergence

Now we can turn to the issue of the uniform convergence of the Fourier series for piecewise smooth, $2L$-periodic functions.

Theorem

Assume that $f(x)$ is continuous on $[-L, L]$ with $f(-L) = f(L)$ and $f'(x)$ is piecewise continuous on $[-L, L]$, then the Fourier series of $f(x)$ converges uniformly to $f(x)$ on $[-L, L]$.

Differentiation of Fourier Series

Recall the theorem:

Theorem

Assume that $f(x)$ is continuous on $(-L, L)$ with $f(-L+) = f(L-)$, and $f'(x)$ is piecewise continuous on $(-L, L)$. Then the Fourier series of $f(x)$ can be differentiated term-by-term.

Remark: this theorem does not depend on the uniform convergence of the Fourier series.

Fourier Coefficients of $f'(x)$

Suppose

$$f'(x) \sim \sum_{n=1}^{\infty} \left(\alpha_n \cos \frac{n\pi x}{L} + \beta_n \sin \frac{n\pi x}{L} \right),$$

then

$$\alpha_n = \frac{n\pi b_n}{L}$$
$$\beta_n = -\frac{n\pi a_n}{L}$$

where a_n and b_n are the Fourier coefficients for $f(x)$ on $[-L, L]$.

Trigonometric Relationship

Suppose we write

$$\begin{aligned} A\cos\theta + B\sin\theta &= R\cos(\theta+\delta) \\ &= R\cos\delta\cos\theta - R\sin\delta\sin\theta \\ A &= R\cos\delta \\ B &= -R\sin\delta \end{aligned}$$

which implies

$$\begin{aligned} R^2 &= A^2 + B^2 \\ \delta &= -\tan^{-1}(B/A). \end{aligned}$$

Furthermore

$$|A\cos\theta + B\sin\theta| \le R$$

for all θ.

Bound

Consider

$$\begin{aligned} |f(x) - S_N(x)| &= \left| \sum_{k=N+1}^{\infty} \left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L} \right) \right| \\ &\le \sum_{k=N+1}^{\infty} \left| a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L} \right| \\ &\le \sum_{k=N+1}^{\infty} \sqrt{a_k^2 + b_k^2} \\ &= \sum_{k=N+1}^{\infty} \sqrt{\frac{L^2}{k^2\pi^2}} \sqrt{\alpha_k^2 + \beta_k^2} \end{aligned}$$

Cauchy-Schwarz Inequality

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)$$

Application of the Cauchy-Schwarz inequality to

$$\begin{aligned}
|f(x) - S_N(x)|^2 &\leq \left(\sum_{k=N+1}^{\infty} \sqrt{\frac{L^2}{k^2\pi^2}}\sqrt{\alpha_k^2 + \beta_k^2}\right)^2 \\
&\leq \left(\sum_{k=N+1}^{\infty} \frac{L^2}{k^2\pi^2}\right)\left(\sum_{k=N+1}^{\infty} (\alpha_k^2 + \beta_k^2)\right) \\
|f(x) - S_N(x)| &\leq \left(\sum_{k=N+1}^{\infty} \frac{L^2}{k^2\pi^2}\right)^{1/2}\left(\sum_{k=N+1}^{\infty} (\alpha_k^2 + \beta_k^2)\right)^{1/2} \\
&= \frac{L}{\pi}\left(\sum_{k=N+1}^{\infty} \frac{1}{k^2}\right)^{1/2}\left(\sum_{k=N+1}^{\infty} (\alpha_n^2 + \beta_n^2)\right)^{1/2}.
\end{aligned}$$

Bound for *p*-Series

Using an improper integral,

$$\sum_{k=N+1}^{\infty} \frac{1}{k^2} \leq \int_{N+1}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{N}.$$

Thus

$$\begin{aligned}
|f(x) - S_N(x)| &\leq \frac{L}{\pi}\left(\sum_{k=N+1}^{\infty} \frac{1}{k^2}\right)^{1/2}\left(\sum_{k=N+1}^{\infty} (\alpha_n^2 + \beta_n^2)\right)^{1/2} \\
&\leq \frac{L}{\pi\sqrt{N}}\left(\sum_{k=N+1}^{\infty} (\alpha_n^2 + \beta_n^2)\right)^{1/2} \\
&\leq \frac{\sqrt{L}}{\pi\sqrt{N}}\left(\int_{-L}^{L} (f'(x))^2\,dx\right)^{1/2}
\end{aligned}$$

by Bessel's inequality. This inequality holds for all $x \in [-L, L]$ and thus the Fourier series for f converges uniformly to f.

Homework

- Read Section 3.11–3.12
- Exercises: 21–25