

# Dirichlet Convergence Theorem

## *Partial Differential Equations*

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# Objectives

In this lesson we will sketch proofs of

- ▶ the Dirichlet Convergence Theorem for Fourier series, and
- ▶ the theorem regarding the uniform convergence of Fourier series.

# Dirichlet Convergence Theorem

## Theorem

*Assume that  $f(x)$  is a piecewise smooth function on the interval  $[-L, L]$  extended to  $(-\infty, \infty)$  periodically with period  $2L$ . Then the Fourier series of  $f(x)$  converges for all  $x$  to the value*

$$\frac{1}{2} (f(x+) + f(x-)).$$

# First Necessary Lemma

In order to prove Dirichlet's convergence theorem for Fourier series we will need the following lemma.

## Lemma

*For any fixed  $L > 0$  and piecewise continuous function  $g(x)$  on interval  $[0, L]$ ,*

$$\lim_{N \rightarrow \infty} \int_0^L g(t) \sin \frac{(2N+1)\pi t}{2L} dt = 0.$$

# Proof

Let  $g(x)$  be piecewise continuous on  $[0, L]$ .

$$\begin{aligned} & \int_0^L g(t) \sin \frac{(2N+1)\pi t}{L} dt \\ &= \int_0^L \left[ g(t) \sin \frac{\pi t}{L} \right] \cos \frac{2N\pi t}{L} dt + \int_0^L \left[ g(t) \cos \frac{\pi t}{L} \right] \sin \frac{2N\pi t}{L} dt \end{aligned}$$

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Consider the even  $2L$ -periodic extension of  $g(t) \sin(\pi t/L)$  and the  $2N$ th coefficient in the Fourier cosine series for this extension.

$$\begin{aligned} a_{2N} &= \frac{2}{L} \int_0^L g(t) \sin \frac{\pi t}{L} \cos \frac{2N\pi t}{L} dt \\ \frac{L}{2} a_{2N} &= \int_0^L g(t) \sin \frac{\pi t}{L} \cos \frac{2N\pi t}{L} dt \\ \lim_{N \rightarrow \infty} \frac{L}{2} a_{2N} &= \lim_{N \rightarrow \infty} \int_0^L g(t) \sin \frac{\pi t}{L} \cos \frac{2N\pi t}{L} dt = 0 \end{aligned}$$

In a similar manner the other integral converges to 0 as  $N \rightarrow \infty$ .

## Nth Partial Sum Revisited

$$\begin{aligned} S_N(x) &= \frac{a_0}{2} + \sum_{n=1}^N \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \\ &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^N \left[ \int_{-L}^L f(t) \cos \frac{n\pi t}{L} dt \right] \cos \frac{n\pi x}{L} \\ &\quad + \frac{1}{L} \sum_{n=1}^N \left[ \int_{-L}^L f(t) \sin \frac{n\pi t}{L} dt \right] \sin \frac{n\pi x}{L} \\ &= \frac{1}{2L} \int_{-L}^L f(t) dt \\ &\quad + \frac{1}{L} \sum_{n=1}^N \int_{-L}^L f(t) \left[ \cos \frac{n\pi t}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi t}{L} \sin \frac{n\pi x}{L} \right] dt \\ &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^N \int_{-L}^L f(t) \cos \frac{n\pi(t-x)}{L} dt \end{aligned}$$

# Dirichlet Kernel

$$\begin{aligned} S_N(x) &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^N \int_{-L}^L f(t) \cos \frac{n\pi(t-x)}{L} dt \\ &= \frac{1}{L} \int_{-L}^L f(t) \left[ \frac{1}{2} + \sum_{n=1}^N \cos \frac{n\pi(t-x)}{L} \right] dt \\ &= \frac{1}{L} \int_{-L}^L f(t) D_N(t-x) dt \end{aligned}$$

where

$$D_N(z) = \frac{1}{2} + \sum_{n=1}^N \cos \frac{n\pi z}{L}$$

is called the **Dirichlet Kernel**.

## Closed Form of Dirichlet Kernel (1 of 3)

$$D_N(z) = \frac{1}{2} + \sum_{n=1}^N \cos \frac{n\pi z}{L}$$

$$\begin{aligned} D_N(z) \sin \frac{\pi z}{L} &= \frac{1}{2} \sin \frac{\pi z}{L} + \sum_{n=1}^N \cos \frac{n\pi z}{L} \sin \frac{\pi z}{L} \\ &= \frac{1}{2} \sin \frac{\pi z}{L} + \frac{1}{2} \sum_{n=1}^N \left[ \sin \frac{(n+1)\pi z}{L} - \sin \frac{(n-1)\pi z}{L} \right] \end{aligned}$$

according to a product-to-sum formula. The summation telescopes to produce

$$D_N(z) \sin \frac{\pi z}{L} = \frac{1}{2} \left[ \sin \frac{(N+1)\pi z}{L} + \sin \frac{N\pi z}{L} \right].$$

## Closed Form of Dirichlet Kernel (2 of 3)

$$\begin{aligned} D_N(z) \sin \frac{\pi z}{L} &= \frac{1}{2} \left[ \sin \frac{(N+1)\pi z}{L} + \sin \frac{N\pi z}{L} \right] \\ &= \frac{1}{2} \left[ \sin \frac{2(N+1)\pi z}{2L} + \sin \frac{2N\pi z}{2L} \right] \\ &= \frac{1}{2} \left[ \sin \left( \frac{(2N+1)\pi z}{2L} + \frac{\pi z}{2L} \right) + \sin \left( \frac{(2N+1)\pi z}{2L} - \frac{\pi z}{2L} \right) \right] \\ &= \sin \frac{(2N+1)\pi z}{2L} \cos \frac{\pi z}{2L} \end{aligned}$$

using a product-to-sum formula.

## Closed Form of Dirichlet Kernel (3 of 3)

$$\begin{aligned} D_N(z) \sin \frac{\pi Z}{L} &= \sin \frac{(2N+1)\pi Z}{2L} \cos \frac{\pi Z}{2L} \\ D_N(z) &= \frac{\sin \frac{(2N+1)\pi Z}{2L} \cos \frac{\pi Z}{2L}}{\sin \frac{\pi Z}{L}} \quad (\text{if } z \neq kL \text{ with } k \in \mathbb{Z}) \\ &= \frac{\sin \frac{(2N+1)\pi Z}{2L} \cos \frac{\pi Z}{2L}}{\sin \frac{2\pi Z}{2L}} \\ &= \frac{\sin \frac{(2N+1)\pi Z}{2L} \cos \frac{\pi Z}{2L}}{2 \sin \frac{\pi Z}{2L} \cos \frac{\pi Z}{2L}} \\ D_N(z) &= \frac{\sin \frac{(2N+1)\pi Z}{2L}}{2 \sin \frac{\pi Z}{2L}} \end{aligned}$$

using the double-angle formula for the sine and assuming  $z \neq 2kL$ .

## Remarks (1 of 2)

Consider the following limit:

$$\begin{aligned}\lim_{z \rightarrow 2kL} D_N(z) &= \lim_{z \rightarrow 2kL} \frac{\sin \frac{(2N+1)\pi z}{2L}}{2 \sin \frac{\pi z}{2L}} \\&= \lim_{z \rightarrow 2kL} \frac{\frac{(2N+1)\pi}{2L} \cos \frac{(2N+1)\pi z}{2L}}{\frac{\pi}{L} \cos \frac{\pi z}{2L}} \\&= \lim_{z \rightarrow 2kL} \frac{(2N+1) \cos \frac{(2N+1)\pi z}{2L}}{2 \cos \frac{\pi z}{2L}} \\&= \frac{2N+1}{2}\end{aligned}$$

Thus the limit of the Dirichlet Kernel exists as  $z \rightarrow 2kL$ .

## Remarks (2 of 2)

- It can also be shown that

$$\lim_{z \rightarrow (2k+1)L} D_N(z) = \frac{(-1)^N}{2}$$

and thus the closed form of the Dirichlet Kernel is defined for all real numbers.

- Integrating the original definition of the Dirichlet Kernel over  $[0, L]$  yields

$$\int_0^L D_N(z) dz = \frac{L}{2}.$$

- The Dirichlet Kernel is a  $2L$ -periodic and even function.

# Second Necessary Lemma

## Lemma

If  $g(x)$  is a piecewise smooth function on  $[0, L]$ , then

$$\lim_{N \rightarrow \infty} \int_0^L g(t) D_N(t) dt = \frac{L}{2} g(0+).$$

# Return to Proof of Dirichlet Convergence Theorem

Recall,

$$S_N(x) = \frac{1}{L} \int_{-L}^L f(t) D_N(t-x) dt.$$

- ▶ Since  $f(t)D_N(t-x)$  is  $2L$ -periodic, then

$$S_N(x) = \frac{1}{L} \int_{x-L}^{x+L} f(t) D_N(t-x) dt.$$

- ▶ Make the substitution  $s = t - x$ ,

$$\begin{aligned} S_N(x) &= \frac{1}{L} \int_{-L}^L f(x+s) D_N(s) ds \\ &= \frac{1}{L} \int_{-L}^0 f(x+s) D_N(s) ds + \frac{1}{L} \int_0^L f(x+s) D_N(s) ds \\ &= \frac{1}{L} \int_0^L f(x-s) D_N(s) ds + \frac{1}{L} \int_0^L f(x+s) D_N(s) ds. \end{aligned}$$

# Proof

Define,

$$I_N(x) = \frac{1}{L} \int_0^L f(x-s) D_N(s) ds$$

$$J_N(x) = \frac{1}{L} \int_0^L f(x+s) D_N(s) ds.$$

By the second necessary lemma,

$$\lim_{N \rightarrow \infty} I_N(x) = \lim_{N \rightarrow \infty} \frac{1}{L} \int_0^L f(x-s) D_N(s) ds = \frac{1}{L} \cdot \frac{L}{2} f(x-) = \frac{1}{2} f(x-)$$

$$\lim_{N \rightarrow \infty} J_N(x) = \lim_{N \rightarrow \infty} \frac{1}{L} \int_0^L f(x+s) D_N(s) ds = \frac{1}{L} \cdot \frac{L}{2} f(x+) = \frac{1}{2} f(x+),$$

and thus

$$\lim_{N \rightarrow \infty} S_N(x) = \frac{1}{2} [f(x-) + f(x+)].$$

# Fourier Series Uniform Convergence

Now we can turn to the issue of the uniform convergence of the Fourier series for piecewise smooth,  $2L$ -periodic functions.

## Theorem

*Assume that  $f(x)$  is continuous on  $[-L, L]$  with  $f(-L) = f(L)$  and  $f'(x)$  is piecewise continuous on  $[-L, L]$ , then the Fourier series of  $f(x)$  converges uniformly to  $f(x)$  on  $[-L, L]$ .*

# Differentiation of Fourier Series

Recall the theorem:

## Theorem

*Assume that  $f(x)$  is continuous on  $(-L, L)$  with  $f(-L+) = f(L-)$ , and  $f'(x)$  is piecewise continuous on  $(-L, L)$ . Then the Fourier series of  $f(x)$  can be differentiated term-by-term.*

# Differentiation of Fourier Series

Recall the theorem:

## Theorem

*Assume that  $f(x)$  is continuous on  $(-L, L)$  with  $f(-L+) = f(L-)$ , and  $f'(x)$  is piecewise continuous on  $(-L, L)$ . Then the Fourier series of  $f(x)$  can be differentiated term-by-term.*

**Remark:** this theorem does not depend on the uniform convergence of the Fourier series.

# Fourier Coefficients of $f'(x)$

Suppose

$$f'(x) \sim \sum_{n=1}^{\infty} \left( \alpha_n \cos \frac{n\pi x}{L} + \beta_n \sin \frac{n\pi x}{L} \right),$$

then

$$\alpha_n = \frac{n\pi b_n}{L}$$
$$\beta_n = -\frac{n\pi a_n}{L}$$

where  $a_n$  and  $b_n$  are the Fourier coefficients for  $f(x)$  on  $[-L, L]$ .

# Trigonometric Relationship

Suppose we write

$$\begin{aligned} A \cos \theta + B \sin \theta &= R \cos(\theta + \delta) \\ &= R \cos \delta \cos \theta - R \sin \delta \sin \theta \end{aligned}$$

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which implies

$$\begin{aligned}R^2 &= A^2 + B^2 \\ \delta &= -\tan^{-1}(B/A).\end{aligned}$$

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which implies

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Furthermore

$$|A \cos \theta + B \sin \theta| \leq R$$

for all  $\theta$ .

# Bound

Consider

$$|f(x) - S_N(x)| = \left| \sum_{k=N+1}^{\infty} \left( a_k \cos \frac{k\pi x}{L} + b_k \sin \frac{k\pi x}{L} \right) \right|$$

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# Cauchy-Schwarz Inequality

$$\left( \sum_{k=1}^n x_k y_k \right)^2 \leq \left( \sum_{k=1}^n x_k^2 \right) \left( \sum_{k=1}^n y_k^2 \right)$$

Application of the Cauchy-Schwarz inequality to

$$\begin{aligned} |f(x) - S_N(x)|^2 &\leq \left( \sum_{k=N+1}^{\infty} \sqrt{\frac{L^2}{k^2 \pi^2}} \sqrt{\alpha_k^2 + \beta_k^2} \right)^2 \\ &\leq \left( \sum_{k=N+1}^{\infty} \frac{L^2}{k^2 \pi^2} \right) \left( \sum_{k=N+1}^{\infty} (\alpha_k^2 + \beta_k^2) \right) \end{aligned}$$

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# Bound for $p$ -Series

Using an improper integral,

$$\sum_{k=N+1}^{\infty} \frac{1}{k^2} \leq \int_{N+1}^{\infty} \frac{1}{x^2} dx = \frac{1}{N}.$$

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Thus

$$\begin{aligned} |f(x) - S_N(x)| &\leq \frac{L}{\pi} \left( \sum_{k=N+1}^{\infty} \frac{1}{k^2} \right)^{1/2} \left( \sum_{k=N+1}^{\infty} (\alpha_n^2 + \beta_n^2) \right)^{1/2} \\ &\leq \frac{L}{\pi\sqrt{N}} \left( \sum_{k=N+1}^{\infty} (\alpha_n^2 + \beta_n^2) \right)^{1/2} \\ &\leq \frac{\sqrt{L}}{\pi\sqrt{N}} \left( \int_{-L}^L (f'(x))^2 dx \right)^{1/2} \end{aligned}$$

by Bessel's inequality. This inequality holds for all  $x \in [-L, L]$  and thus the Fourier series for  $f$  converges uniformly to  $f$ .