

# Convergence of Fourier Series

MATH 467 *Partial Differential Equations*

J Robert Buchanan

Department of Mathematics

Fall 2022

# Objectives

In this lesson we will explore the questions:

- ▶ What are the conditions that guarantee the Fourier series of a given function  $f(x)$  converges?
- ▶ If the Fourier series of a given function  $f(x)$  converges, does it converge to the value of  $f(x)$  at a given  $x$ ?

## Short Answer to First Question

A Fourier series will converge for a large class of functions, though we will prove convergence only for the class of **piecewise smooth** functions.

### Definition

A function  $f(x)$  is **piecewise continuous** on  $[a, b]$  (or  $(a, b)$ ) if there are finitely many points  $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = b$ , such that

- ▶  $f(x)$  is continuous on  $(x_{i-1}, x_i)$  for all  $i = 1, 2, \dots, n$ ,
- ▶ the one-sided limits  $\lim_{x \rightarrow x_i^-} f(x)$  and  $\lim_{x \rightarrow x_i^+} f(x)$  exist for all  $i = 1, 2, \dots, n-1$ , and
- ▶ the one-sided limits  $\lim_{x \rightarrow a^+} f(x)$  and  $\lim_{x \rightarrow b^-} f(x)$  both exist.

# Remarks

- ▶ The limits mentioned in the definition of piecewise continuous must be finite, real numbers.
- ▶ Function  $f(x)$  is piecewise continuous on  $(-\infty, \infty)$  if it is piecewise continuous on every finite interval  $[a, b]$ .
- ▶ Function  $f(x)$  is **piecewise smooth** on  $[a, b]$  if  $f'(x)$  is piecewise continuous on  $[a, b]$ .
- ▶ Since any piecewise continuous function is integrable, the Fourier coefficients of any piecewise continuous function are well-defined (but this is not enough to show the Fourier series converges).

# Notation

For any  $c \in [a, b]$ , define

$$\lim_{x \rightarrow c^+} f(x) = f(c+) \quad \text{and} \quad \lim_{x \rightarrow c^-} f(x) = f(c-).$$

Function  $f(x)$  is continuous at  $c$  if and only if  $f(c+) = f(c-) = f(c)$ .

## Examples

Determine which of the following functions is piecewise smooth, piecewise continuous, or neither on  $[-L, L]$  for some  $L > 0$ .

$$f_1(x) = \begin{cases} L & \text{if } -L \leq x < 0 \\ x & \text{if } 0 \leq x \leq L \end{cases}$$

$$f_2(x) = \begin{cases} x^2 & \text{if } -L \leq x < 0 \\ x - 1 & \text{if } 0 \leq x \leq L \end{cases}$$

$$f_3(x) = \cos \frac{1}{x}$$

$$f_4(x) = x \cos \frac{1}{x}$$

$$f_5(x) = x^{3/5}$$

$$f_6(x) = x^{6/5}$$

$$f_7(x) = \frac{1}{x}$$

## Discussion (1 of 7)

$$f_1(x) = \begin{cases} L & \text{if } -L \leq x < 0 \\ x & \text{if } 0 \leq x \leq L \end{cases}$$

Function  $f_1(x)$  is piecewise continuous and piecewise smooth on  $[-L, L]$  with

$$f_1(-L+) = f_1(0-) = f_1(L-) = L$$

$$f_1(0+) = 0$$

$$f'_1(x) = \begin{cases} 0 & \text{if } -L < x < 0 \\ 1 & \text{if } 0 < x < L \end{cases}$$

$$f'_1(-L+) = f'_1(0-) = 0$$

$$f'_1(0+) = f'_1(L-) = 1.$$

## Discussion (2 of 7)

$$f_2(x) = \begin{cases} x^2 & \text{if } -L \leq x < 0 \\ x - 1 & \text{if } 0 \leq x \leq L \end{cases}$$

Function  $f_2(x)$  is piecewise continuous and piecewise smooth on  $[-L, L]$  with

$$f_2(-L+) = L^2$$

$$f_2(0-) = 0$$

$$f_2(0+) = -1$$

$$f_2(L-) = L - 1$$

$$f_2'(x) = \begin{cases} 2x & \text{if } -L < x < 0 \\ 1 & \text{if } 0 < x < L \end{cases}$$

$$f_2'(-L+) = -2L$$

$$f_2'(0-) = 0$$

$$f_2'(0+) = f_2'(L-) = 1.$$

## Discussion (3 of 7)

$$f_3(x) = \cos \frac{1}{x}$$

Function  $f_3(x)$  is neither piecewise continuous nor piecewise smooth on  $[-L, L]$  since  $f_3(0-)$  does not exist.

$$f'_3(x) = \frac{1}{x^2} \sin \frac{1}{x}$$

Note that  $f'_3(0-)$  does not exist.

## Discussion (4 of 7)

$$f_4(x) = x \cos \frac{1}{x}$$

Function  $f_4(x)$  is piecewise continuous but not piecewise smooth on  $[-L, L]$ .

$$f_4(-L+) = -L \cos \frac{1}{L}$$

$$f_4(0-) = f_4(0+) = 0$$

$$f_4(L-) = L \cos \frac{1}{L}$$

$$f_4'(x) = \cos \frac{1}{x} - \frac{1}{x} \sin \frac{1}{x}$$

Note that  $f_4'(0-)$  does not exist.

## Discussion (5 of 7)

$$f_5(x) = x^{3/5}$$

Function  $f_5(x)$  is continuous but not piecewise smooth on  $[-L, L]$ .

$$f'_5(x) = \frac{3}{5x^{2/5}}$$

Note that  $f'_5(0-)$  does not exist.

## Discussion (6 of 7)

$$f_6(x) = x^{6/5}$$

Function  $f_6(x)$  is continuous and piecewise smooth on  $[-L, L]$ .

$$f'_6(x) = \frac{6}{5}x^{1/5}$$

## Discussion (7 of 7)

$$f_7(x) = \frac{1}{x}$$

Function  $f_7(x)$  is neither piecewise continuous nor piecewise smooth on  $[-L, L]$ . Note that  $f_7(0-)$  does not exist.

$$f_7'(x) = -\frac{1}{x^2}$$

Note that  $f_7'(0-)$  does not exist.

# Dirichlet Convergence Theorem

## Theorem

*Assume that  $f(x)$  is a piecewise smooth function on the interval  $[-L, L]$  extended to  $(-\infty, \infty)$  periodically with period  $2L$ . Then the Fourier series of  $f(x)$  converges for all  $x$  to the value*

$$\frac{1}{2} (f(x+) + f(x-)) .$$

## Remarks:

- ▶ If  $f$  is continuous at  $x = x_0$  then the Fourier series converges to  $f(x_0)$  when  $x = x_0$ .
- ▶ If  $f$  has a jump or removable discontinuity at  $x = x_0$ , the Fourier series converges to the average of the limits of  $f$  from the left and right at  $x = x_0$ .

## Example

Consider the piecewise-defined function

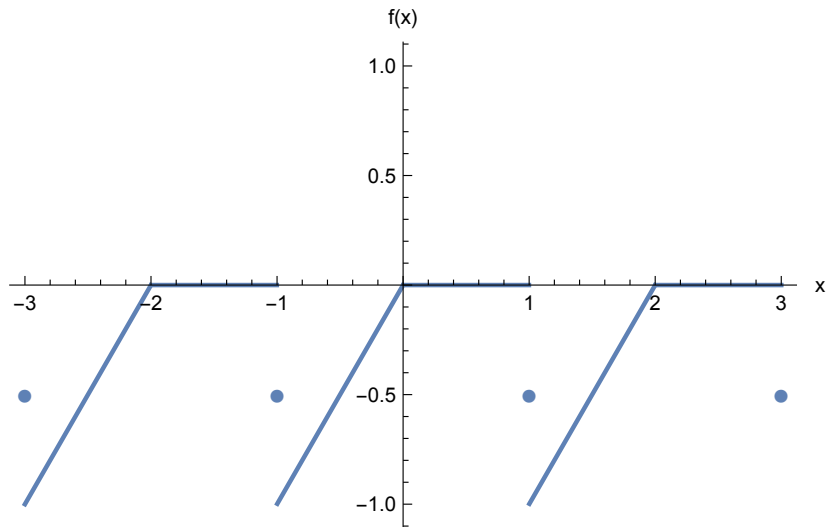
$$f(x) = \begin{cases} x & \text{if } -1 \leq x < 0, \\ 0 & \text{if } 0 \leq x < 1. \end{cases}$$

Its Fourier series representation is

$$f(x) \sim -\frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(n\pi x) + \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2\pi^2} \cos((2n-1)\pi x).$$

Sketch the graph of the Fourier series.

# Graph



# Application: Finding the Sum of a Series

If an infinite series is made up of Fourier coefficients for some function, the function can be used to sum the series.

## Example

1. Find the Fourier series for  $f(x) = |x|$  on  $[-\pi, \pi]$ .
2. Use the Fourier series and  $f(x)$  to find

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

## Solution

Function  $f(x) = |x|$  is an even function.

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) \, dx = \begin{cases} -4/(n^2\pi) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Fourier series:

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)x)$$

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)(0))$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

## Application: Finding the Sum of a Series

Consider the function  $f(x) = \begin{cases} -1 & \text{if } -\pi \leq x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \leq \pi. \end{cases}$

1. Find the Fourier series for  $f(x)$  on  $[-\pi, \pi]$ .
2. Sketch the graph of the Fourier series for  $f(x)$ .
3. Use the Fourier series and  $f(x)$  to find

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1}.$$

## Solution

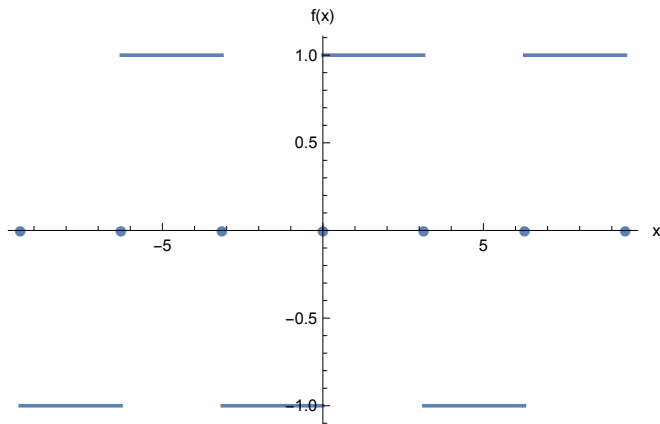
Function  $f(x)$  is an odd function.

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \\ &= \frac{-1}{\pi} \int_{-\pi}^0 \sin(nx) dx + \frac{1}{\pi} \int_0^{\pi} \sin(nx) dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin(nx) dx \\ &= \frac{2}{n\pi} (1 - \cos(n\pi)) \\ &= \begin{cases} 4/(n\pi) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Fourier series:

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)x).$$

# Graph



**Question:** why does the Fourier series converge to  $f(x)$  on  $(-\pi, \pi)$ ?

# Summing the Series

Let  $x = \pi/2$ .

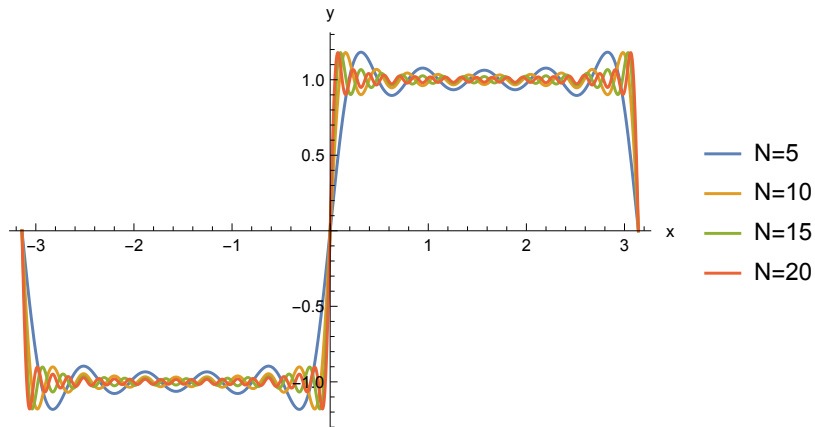
$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)x)$$

$$1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left((2n-1)\frac{\pi}{2}\right)$$

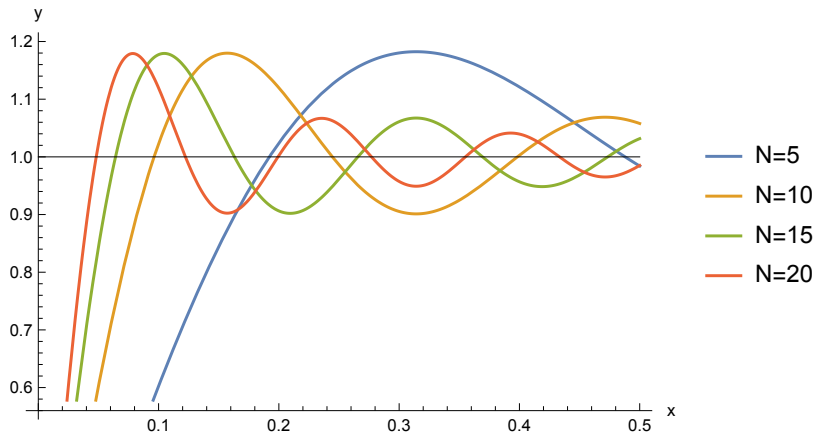
$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$$

$$-\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$

# Partial Sums



## Detailed View



# Gibbs Phenomenon

- ▶ Oscillation of the partial sums of a Fourier series near a jump discontinuity is called the **Gibbs phenomenon**.
- ▶ It was first mathematically explained by Josiah Willard Gibbs, though others had considered it (including Albert A. Michelson of the Michelson-Morley experiment).
- ▶ We will outline an explanation of the Gibbs phenomenon.

# Partial Sum

Denote the  $N$ th partial sum of the Fourier series as

$$s_N(x) = \frac{4}{\pi} \sum_{n=1}^N \frac{1}{2n-1} \sin((2n-1)x).$$

A calculus argument can locate the maxima in the graph of  $s_N(x)$  near  $x = 0$ .

$$s'_N(x) = \frac{4}{\pi} \sum_{n=1}^N \cos((2n-1)x)$$

$$(\sin x)s'_N(x) = \frac{4}{\pi} \sum_{n=1}^N \sin x \cos((2n-1)x)$$

Use a product-to-sum formula on the right-hand side.

## Critical Numbers

$$\begin{aligned}(\sin x)s'_N(x) &= \frac{4}{\pi} \sum_{n=1}^N \cos((2n-1)x) \sin x \\&= \frac{2}{\pi} \sum_{n=1}^N (\sin(2nx) - \sin(2(n-1)x)) \\&= \frac{2}{\pi} \sin(2Nx)\end{aligned}$$

If  $0 < x < \pi$  then  $s'_N(x) = 0$  if and only if  $\sin(2Nx) = 0$ . Thus the critical numbers in  $(0, \pi)$  are

$$x = \frac{m\pi}{2N} \quad \text{for } m = 1, 2, \dots, 2N-1.$$

## Maxima or Minima?

Apply the Second Derivative Test to determine whether the critical numbers are maxima or minima.

$$\begin{aligned}(\sin x)s'_N(x) &= \frac{2}{\pi} \sin(2Nx) \\ (\cos x)s'_N(x) + (\sin x)s''_N(x) &= \frac{4N}{\pi} \cos(2Nx)\end{aligned}$$

Let  $x = \pi/(2N)$ , the critical number closest to  $x = 0$ .

$$\begin{aligned}\left(\cos \frac{\pi}{2N}\right) s'_N\left(\frac{\pi}{2N}\right) + \left(\sin \frac{\pi}{2N}\right) s''_N\left(\frac{\pi}{2N}\right) &= \frac{4N}{\pi} \cos \pi \\ \left(\sin \frac{\pi}{2N}\right) s''_N\left(\frac{\pi}{2N}\right) &= -\frac{4N}{\pi}\end{aligned}$$

This implies  $s_N(\pi/2N)$  is a local maximum.

## Value of Local Maximum

$$\begin{aligned} s_N\left(\frac{\pi}{2N}\right) &= \frac{4}{\pi} \sum_{n=1}^N \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi}{2N}\right) \\ &= \frac{2}{\pi} \sum_{n=1}^N \frac{2N\pi}{(2n-1)N\pi} \sin\left(\frac{(2n-1)\pi}{2N}\right) \\ &= \frac{2}{\pi} \sum_{n=1}^N \frac{2N}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi}{2N}\right) \frac{\pi}{N} \end{aligned}$$

**Remark:** the term inside the summation is a Riemann sum with  $\Delta x = \pi/N$ ,  $w_n = (2n-1)\pi/(2N)$ , and  $f(x) = \frac{1}{x} \sin x$ .

## Limit of Riemann Sum

$$\begin{aligned}\lim_{N \rightarrow \infty} s_N \left( \frac{\pi}{2N} \right) &= \lim_{N \rightarrow \infty} \frac{2}{\pi} \sum_{n=1}^N \frac{2N}{(2n-1)\pi} \sin \left( \frac{(2n-1)\pi}{2N} \right) \frac{\pi}{N} \\ &= \frac{2}{\pi} \int_0^{\pi} \frac{\sin x}{x} dx \\ &\approx 1.179\end{aligned}$$

**Remark:** thus no matter how many terms are included in the Fourier series there is an  $x \rightarrow 0^+$  for which  $f(x) \approx 1.179 > 1$ .

# Pointwise vs. Uniform Convergence

## Definition

Suppose  $\{f_n\}_{n=1}^{\infty}$  is a sequence of functions defined on domain  $D$  and the sequence of values  $\{f_n(x)\}_{n=1}^{\infty}$  converges for each  $x \in S \subset D$ . Then  $\{f_n\}_{n=1}^{\infty}$  is said to **converge pointwise on  $S$  to  $f$**  defined by

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \text{for } x \in S.$$

While the Fourier series converges **pointwise** to  $f(x)$  it does not converge **uniformly** to  $f(x)$ .

# Uniform Convergence

## Definition

A sequence of functions  $\{f_n\}_{n=1}^{\infty}$  defined on domain  $D$  **converges uniformly to  $f$  on  $D$**  provided there exists a sequence of positive real numbers  $\{\epsilon_n\}_{n=1}^{\infty}$  for which  $\lim_{n \rightarrow \infty} \epsilon_n = 0$  and

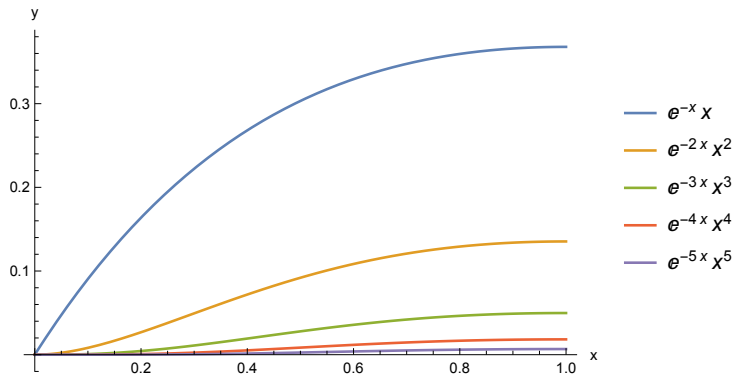
$$|f_n(x) - f(x)| < \epsilon_n$$

for all  $n \in \mathbb{N}$  and for all  $x \in D$ .

**Remark:** an infinite series converges uniformly if its sequence of partial sums converges uniformly.

# Illustration

Sequence  $\{x^n e^{-nx}\}_{n=1}^{\infty}$  converges uniformly to  $f(x) = 0$  on  $D = [0, \infty)$ .



# Justification

- ▶ Let  $f_n(x) = x^n e^{-nx}$  for  $n \in \mathbb{N}$  and  $x \geq 0$ .
- ▶ By l'Hôpital's Rule  $\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{x^n}{e^{nx}} = 0$  for all  $x \geq 0$ , thus  $f_n(x) \rightarrow 0$  pointwise for all  $x \geq 0$ . Define  $f(x) = 0$ .
- ▶ The First Derivative Test shows that  $0 \leq f_n(x) \leq f_n(1) = e^{-n}$  for all  $x \geq 0$ , so let  $\epsilon_n = e^{-n}$  and note that  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ .
- ▶ The sequence of functions  $f_n(x) \rightarrow f(x)$  uniformly for  $x \geq 0$  since  $|f_n(x) - f(0)| = f_n(x) \leq \epsilon_n$  for all  $n \in \mathbb{N}$ .

## Examples (1 of 3)

Sequence  $\{x^n\}_{n=1}^{\infty}$  converges pointwise on  $[0, 1]$  to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases} \quad \text{but does not converge uniformly.}$$

# Justification

- ▶ Let  $x \in [0, 1)$  then  $\lim_{n \rightarrow \infty} x^n = 0$ . Also,  $\lim_{n \rightarrow \infty} 1^n = 1$ , so  $x^n \rightarrow f(x)$  pointwise for  $0 \leq x \leq 1$ .
- ▶ Let  $\{\epsilon_n\}_{n=1}^{\infty}$  be any sequence of positive real numbers such that  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ . Without loss of generality assume  $0 < \epsilon_n < 1$  for all  $n \in \mathbb{N}$ .
- ▶ Note that  $0 < 1 - \epsilon_n < 1$  for all  $n \in \mathbb{N}$  and thus  $0 < (1 - \epsilon_n)^{1/n} < 1$  as well. Therefore for all  $n \in \mathbb{N}$  there exists  $x_n$  such that  $(1 - \epsilon_n)^{1/n} < x_n < 1$ .
- ▶ Consider  $x_n^n > 1 - \epsilon_n > 0$  and thus  $x_n^n \not\rightarrow 0$  as  $n \rightarrow \infty$  and the convergence is not uniform.

## Examples (2 of 3)

The sequence  $\{x^n\}_{n=1}^{\infty}$  converges uniformly to  $f(x) = 0$  on  $[0, b]$  for any  $0 < b < 1$ .

# Justification

- ▶ As in the previous example  $x^n \rightarrow 0$  pointwise for all  $0 \leq x \leq b < 1$ .
- ▶ Define  $\epsilon_n = b^n$  for all  $n \in \mathbb{N}$ , then  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ .
- ▶ Note that  $x^n \leq b^n$  for all  $0 \leq x \leq b$ , or equivalently,  $|x^n - 0| \leq \epsilon_n$  for all  $0 \leq x \leq b$ . Hence the convergence is uniform.

## Examples (3 of 3)

The infinite series  $\sum_{n=1}^{\infty} x^n$  converges pointwise on  $(-1, 1)$  to

$$f(x) = \frac{x}{1-x} \text{ but not uniformly.}$$

# Justification

- ▶ The  $N$ th partial sum of the series is

$$f_N(x) = x + x^2 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x} - 1.$$

- ▶ For any  $-1 < x < 1$ ,

$$\lim_{N \rightarrow \infty} f_N(x) = \lim_{N \rightarrow \infty} \left[ \frac{1 - x^{N+1}}{1 - x} - 1 \right] = \frac{1}{1 - x} - 1 = \frac{x}{1 - x} = f(x),$$

(this proves pointwise convergence).

- ▶ Consider

$$\begin{aligned} \lim_{x \rightarrow 1^-} |f_N(x) - f(x)| &= \lim_{x \rightarrow 1^-} \left| \frac{1 - x^{N+1}}{1 - x} - 1 - \frac{x}{1 - x} \right| \\ &= \lim_{x \rightarrow 1^-} \frac{x^{N+1}}{1 - x} = \infty, \end{aligned}$$

so the convergence is not uniform.

# Properties Preserved by Uniform Convergence

Suppose the series  $\sum_{n=1}^{\infty} u_n(x)$  converges uniformly to its sum  $u(x)$  on  $[a, b]$ .

- ▶ If for each  $n$ ,  $u_n(x)$  is continuous on  $[a, b]$ , then the sum  $u(x)$  is continuous on  $[a, b]$ .
- ▶ If for each  $n$ ,  $u_n(x)$  is integrable on  $[a, b]$ , then the sum  $u(x)$  is integrable on  $[a, b]$ , and

$$\int_a^b u(x) dx = \int_a^b \left( \sum_{n=1}^{\infty} u_n(x) \right) dx = \sum_{n=1}^{\infty} \int_a^b u_n(x) dx.$$

- ▶ If for each  $n$ ,  $u'_n(x)$  exists and  $\sum_{n=1}^{\infty} u'_n(x)$  converges uniformly on  $[a, b]$ , then the sum  $u(x)$  is differentiable on  $[a, b]$  and the derivative can be obtained by differentiating the series term by term,

$$u'(x) = \left( \sum_{n=1}^{\infty} u_n(x) \right)' = \sum_{n=1}^{\infty} u'_n(x) \quad \text{for all } x \in [a, b].$$

# Weierstrass $M$ -Test

The following theorem provides a convenient means of determining whether an infinite series converges uniformly.

## Theorem

*Let  $\sum_{n=1}^{\infty} u_n(x)$  be a series of functions defined on an interval  $[a, b]$  and suppose that for each  $n$  there is a non-negative number  $M_n$  such that  $|u_n(x)| \leq M_n$  for all  $x \in [a, b]$  and  $\sum_{n=1}^{\infty} M_n$  converges, then*

*$\sum_{n=1}^{\infty} u_n(x)$  converges uniformly on  $[a, b]$ .*

## Example

Show that  $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$  converges uniformly on  $(-\infty, \infty)$ .

Since  $\frac{1}{n^2} |\cos(nx)| \leq \frac{1}{n^2}$  for all  $x \in \mathbb{R}$  and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, then the series converges uniformly for all  $x \in \mathbb{R}$ .

# Homework

- ▶ Read Sections 3.6–3.7
- ▶ Exercises: 11–19