Convergence of Fourier Series

MATH 467 *Partial Differential Equations*

J. Robert Buchanan

Department of Mathematics

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Objectives

In this lesson we will explore the questions:

- What are the conditions that guarantee the Fourier series of a given function $f(x)$ converges?
- If the Fourier series of a given function $f(x)$ converges, does it converge to the value of $f(x)$ at a given x?

Short Answer to First Question

A Fourier series will converge for a large class of functions, though we will prove convergence only for the class of **piecewise smooth** functions.

Definition

A function $f(x)$ is **piecewise continuous** on $[a, b]$ (or (a, b)) if there are finitely many points $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = b$, such that

- $f(x)$ is continuous on (x_{i-1}, x_i) for all $i = 1, 2, \ldots, n$,
- the one-sided limits $\lim\limits_{x \to x_i^-} f(x)$ and $\lim\limits_{x \to x_i^+} f(x)$ exist for all $i = 1, 2, \ldots, n-1$, and
- the one-sided limits $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to b^-} f(x)$ both exist.

Remarks

- The limits mentioned in the definition of piecewise continuous must be finite, real numbers.
- Function $f(x)$ is piecewise continuous on $(-\infty, \infty)$ if it is piecewise continuous on every finite interval $[a, b]$.
- Function $f(x)$ is **piecewise smooth** on $[a, b]$ if $f'(x)$ is piecewise continuous on $[a, b]$.
- Since any piecewise continuous function is integrable, the Fourier coefficients of any piecewise continuous function are well-defined (but this is not enough to show the Fourier series converges).

Notation

For any $c \in [a, b]$, define

$$\lim_{x \to c^+} f(x) = f(c+) \quad \text{and} \quad \lim_{x \to c^-} f(x) = f(c-).$$

Function $f(x)$ is continuous at c if and only if $f(c+) = f(c-) = f(c)$.

Examples

Determine which of the following functions is piecewise smooth, piecewise continuous, or neither on $[-L, L]$ for some $L > 0$.

$$
\begin{aligned}
f_1(x) &= \begin{cases} L & \text{if } -L \leq x < 0 \\ x & \text{if } 0 \leq x \leq L \end{cases} \\
f_2(x) &= \begin{cases} x^2 & \text{if } -L \leq x < 0 \\ x - 1 & \text{if } 0 \leq x \leq L \end{cases} \\
f_3(x) &= \cos \frac{1}{x} \\
f_4(x) &= x \cos \frac{1}{x} \\
f_5(x) &= x^{3/5} \\
f_6(x) &= x^{6/5} \\
f_7(x) &= \frac{1}{x}
\end{aligned}
$$

Discussion (1 of 7)

$$f_1(x) = \begin{cases} L & \text{if } -L \leq x < 0 \\ x & \text{if } 0 \leq x \leq L \end{cases}$$

Function $f_1(x)$ is piecewise continuous and piecewise smooth on $[-L, L]$ with

$$\begin{aligned} f_1(-L+) &= f_1(0-) = f_1(L-) = L \\ f_1(0+) &= 0 \\ f_1'(x) &= \begin{cases} 0 & \text{if } -L < x < 0 \\ 1 & \text{if } 0 < x < L \end{cases} \\ f_1'(-L+) &= f_1'(0-) = 0 \\ f_1'(0+) &= f_1'(L-) = 1. \end{aligned}$$

Discussion (2 of 7)

$$f_2(x) = \begin{cases} x^2 & \text{if } -L \le x < 0 \\ x - 1 & \text{if } 0 \le x \le L \end{cases}$$

Function $f_2(x)$ is piecewise continuous and piecewise smooth on $[-L, L]$ with

$$\begin{aligned}
f_2(-L+) &= L^2 \\
f_2(0-) &= 0 \\
f_2(0+) &= -1 \\
f_2(L-) &= L - 1 \\
f_2'(x) &= \begin{cases} 2x & \text{if } -L < x < 0 \\ 1 & \text{if } 0 < x < L \end{cases} \\
f_2'(-L+) &= -2L \\
f_2'(0-) &= 0 \\
f_2'(0+) &= f_2'(L-) = 1.
\end{aligned}$$

Discussion (3 of 7)

$$f_3(x) = \cos \frac{1}{x}$$

Function $f_3(x)$ is neither piecewise continuous nor piecewise smooth on $[-L, L]$ since $f_3(0-)$ does not exist.

$$f_3'(x) = \frac{1}{x^2} \sin \frac{1}{x}$$

Note that $f_3'(0-)$ does not exist.

Discussion (4 of 7)

$$f_4(x) = x \cos \frac{1}{x}$$

Function $f_4(x)$ is piecewise continuous but not piecewise smooth on $[-L, L]$.

$$\begin{aligned} f_4(-L+) &= -L \cos \frac{1}{L} \\ f_4(0-) &= f_4(0+) = 0 \\ f_4(L-) &= L \cos \frac{1}{L} \end{aligned}$$

$$f_4'(x) = \cos \frac{1}{x} - \frac{1}{x} \sin \frac{1}{x}$$

Note that $f_4'(0-)$ does not exist.

Discussion (5 of 7)

$$f_5(x) = x^{3/5}$$

Function $f_5(x)$ is continuous but not piecewise smooth on $[-L, L]$.

$$f_5'(x) = \frac{3}{5x^{2/5}}$$

Note that $f_5'(0-)$ does not exist.

Discussion (6 of 7)

$$f_6(x) = x^{6/5}$$

Function $f_6(x)$ is continuous and piecewise smooth on $[-L, L]$.

$$f_6'(x) = \frac{6}{5}x^{1/5}$$

Discussion (7 of 7)

$$f_7(x) = \frac{1}{x}$$

Function $f_7(x)$ is neither piecewise continuous nor piecewise smooth on $[-L, L]$. Note that $f_7(0-)$ does not exist.

$$f_7'(x) = \frac{-1}{x^2}$$

Note that $f_7'(0-)$ does not exist.

Dirichlet Convergence Theorem

Theorem

Assume that $f(x)$ is a piecewise smooth function on the interval $[-L, L]$ extended to $(-\infty, \infty)$ periodically with period $2L$. Then the Fourier series of $f(x)$ converges for all x to the value

$$\frac{1}{2}\left(f(x+) + f(x-)\right).$$

Remarks:

- If f is continuous at $x = x_0$ then the Fourier series converges to $f(x_0)$ when $x = x_0$.
- If f has a jump or removable discontinuity at $x = x_0$, the Fourier series converges to the average of the limits of f from the left and right at $x = x_0$.

Example

Consider the piecewise-defined function

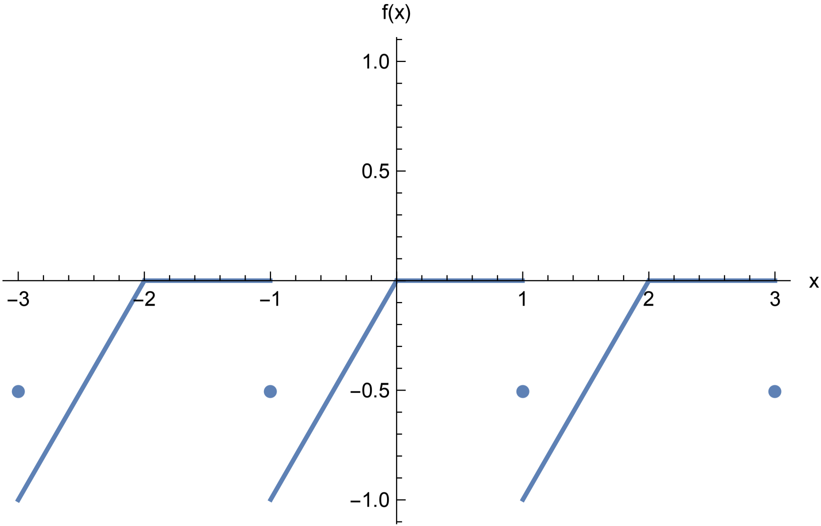
$$f(x) = \begin{cases} x & \text{if } -1 \leq x < 0, \\ 0 & \text{if } 0 \leq x < 1. \end{cases}$$

Its Fourier series representation is

$$\begin{aligned} f(x) \quad \sim \quad & -\frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(n\pi x) \\ & + \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2\pi^2} \cos((2n-1)\pi x). \end{aligned}$$

Sketch the graph of the Fourier series.

Graph



Application: Finding the Sum of a Series

If an infinite series is made up of Fourier coefficients for some function, the function can be used to sum the series.

Example

1. Find the Fourier series for $f(x) = |x|$ on $[-\pi, \pi]$.
2. Use the Fourier series and $f(x)$ to find

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Solution

Function $f(x) = |x|$ is an even function.

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi \\ a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \begin{cases} -4/(n^2\pi) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Fourier series:

$$\begin{aligned} |x| &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos((2n-1)x) \\ 0 &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos((2n-1)(0)) \\ \frac{\pi^2}{8} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \end{aligned}$$

Application: Finding the Sum of a Series

Consider the function $f(x) = \begin{cases} -1 & \text{if } -\pi \leq x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \leq \pi. \end{cases}$

1. Find the Fourier series for $f(x)$ on $[-\pi, \pi]$.
2. Sketch the graph of the Fourier series for $f(x)$.
3. Use the Fourier series and $f(x)$ to find

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1}.$$

Solution

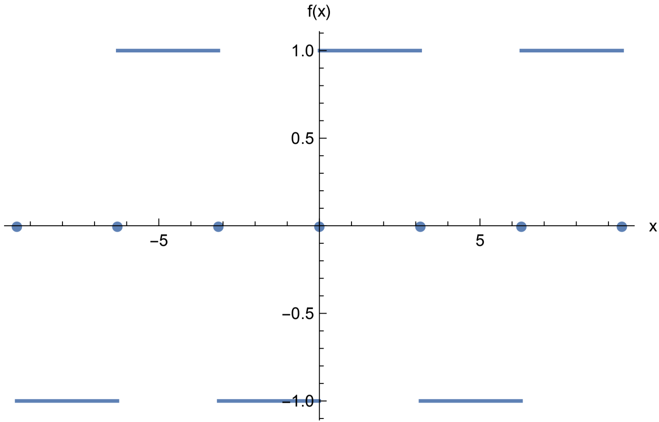
Function $f(x)$ is an odd function.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{-1}{\pi}\int_{-\pi}^{0}\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi}\sin(nx)\,dx \\ &= \frac{2}{n\pi}(1-\cos(n\pi)) \\ &= \begin{cases} 4/(n\pi) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Fourier series:

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x).$$

Graph



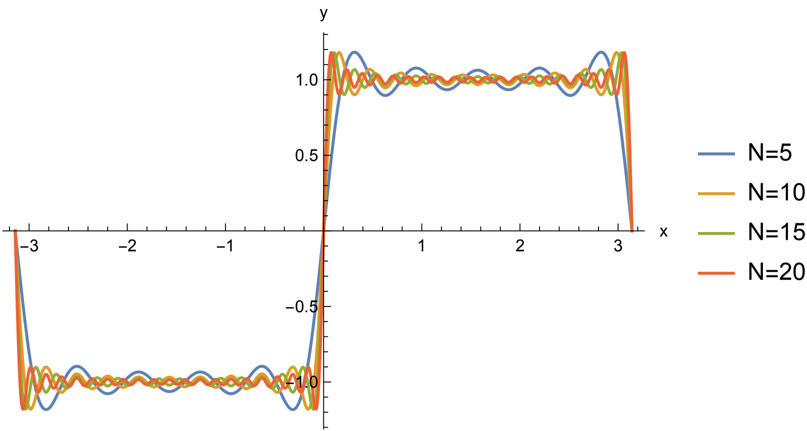
Question: why does the Fourier series converge to $f(x)$ on $(-\pi, \pi)$?

Summing the Series

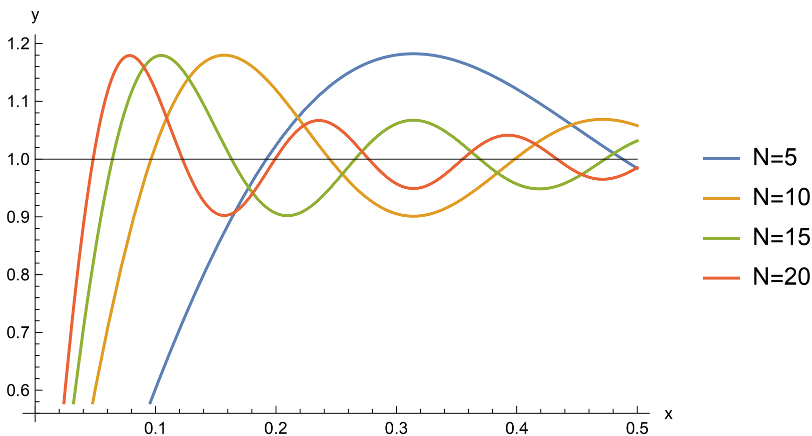
Let $x = \pi/2$.

$$
\begin{aligned}
f(x) &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x) \\
1 &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left((2n-1)\frac{\pi}{2}\right) \\
\frac{\pi}{4} &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} \\
-\frac{\pi}{4} &= \sum_{n=1}^{\infty}\frac{(-1)^{n}}{2n-1}
\end{aligned}
$$

Partial Sums



Detailed View



Gibbs Phenomenon

- Oscillation of the partial sums of a Fourier series near a jump discontinuity is called the **Gibbs phenomenon**.
- It was first mathematically explained by Josiah Willard Gibbs, though others had considered it (including Albert A. Michelson of the Michelson-Morley experiment).
- We will outline an explanation of the Gibbs phenomenon.

Partial Sum

Denote the Nth partial sum of the Fourier series as

$$s_N(x) = \frac{4}{\pi} \sum_{n=1}^{N} \frac{1}{2n-1} \sin((2n-1)x).$$

A calculus argument can locate the maxima in the graph of $s_N(x)$ near $x = 0$.

$$\begin{aligned} s_N'(x) &= \frac{4}{\pi} \sum_{n=1}^{N} \cos((2n-1)x) \\ (\sin x) s_N'(x) &= \frac{4}{\pi} \sum_{n=1}^{N} \sin x \cos((2n-1)x) \end{aligned}$$

Use a product-to-sum formula on the right-hand side.

Critical Numbers

$$
\begin{aligned}
(\sin x)s_N'(x) &= \frac{4}{\pi}\sum_{n=1}^{N}\cos((2n-1)x)\sin x \\
&= \frac{2}{\pi}\sum_{n=1}^{N}(\sin(2nx)-\sin(2(n-1)x)) \\
&= \frac{2}{\pi}\sin(2Nx)
\end{aligned}
$$

If $0 < x < \pi$ then $s_N'(x) = 0$ if and only if $\sin(2Nx) = 0$. Thus the critical numbers in $(0, \pi)$ are

$$x = \frac{m\pi}{2N} \quad \text{for } m = 1, 2, \ldots, 2N-1.$$

Maxima or Minima?

Apply the Second Derivative Test to determine whether the critical numbers are maxima or minima.

$$\begin{aligned} (\sin x) s_N'(x) &= \frac{2}{\pi} \sin(2Nx) \\ (\cos x) s_N'(x) + (\sin x) s_N''(x) &= \frac{4N}{\pi} \cos(2Nx) \end{aligned}$$

Let $x = \pi/(2N)$, the critical number closest to $x = 0$.

$$\begin{aligned} \left(\cos \frac{\pi}{2N}\right) s_N'\left(\frac{\pi}{2N}\right) + \left(\sin \frac{\pi}{2N}\right) s_N''\left(\frac{\pi}{2N}\right) &= \frac{4N}{\pi} \cos \pi \\ \left(\sin \frac{\pi}{2N}\right) s_N''\left(\frac{\pi}{2N}\right) &= -\frac{4N}{\pi} \end{aligned}$$

This implies $s_N(\pi/2N)$ is a local maximum.

Value of Local Maximum

$$
\begin{aligned}
s_N\left(\frac{\pi}{2N}\right) &= \frac{4}{\pi}\sum_{n=1}^{N}\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi}{2N}\right) \\
&= \frac{2}{\pi}\sum_{n=1}^{N}\frac{2N\pi}{(2n-1)N\pi}\sin\left(\frac{(2n-1)\pi}{2N}\right) \\
&= \frac{2}{\pi}\sum_{n=1}^{N}\frac{2N}{(2n-1)\pi}\sin\left(\frac{(2n-1)\pi}{2N}\right)\frac{\pi}{N}
\end{aligned}
$$

Remark: the term inside the summation is a Riemann sum with $\Delta x = \pi/N$, $w_n = (2n-1)\pi/(2N)$, and $f(x) = \frac{1}{x}\sin x$.

Limit of Riemann Sum

$$
\begin{aligned}
\lim_{N\to\infty} s_N\left(\frac{\pi}{2N}\right) &= \lim_{N\to\infty} \frac{2}{\pi} \sum_{n=1}^{N} \frac{2N}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi}{2N}\right) \frac{\pi}{N} \\
&= \frac{2}{\pi} \int_0^{\pi} \frac{\sin x}{x}\, dx \\
&\approx 1.179
\end{aligned}
$$

Remark: thus no matter how many terms are included in the Fourier series there is an $x \to 0^+$ for which $f(x) \approx 1.179 > 1$.

Pointwise *vs.* Uniform Convergence

Definition

Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions defined on domain D and the sequence of values $\{f_n(x)\}_{n=1}^{\infty}$ converges for each $x \in S \subset D$. Then $\{f_n\}_{n=1}^{\infty}$ is said to **converge pointwise on** S **to** f defined by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for } x \in S.$$

While the Fourier series converges **pointwise** to $f(x)$ it does not converge **uniformly** to $f(x)$.

Uniform Convergence

Definition

A sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined on domain D **converges uniformly to** f **on** D provided there exists a sequence of positive real numbers $\{\epsilon_n\}_{n=1}^{\infty}$ for which $\lim_{n\to\infty} \epsilon_n = 0$ and

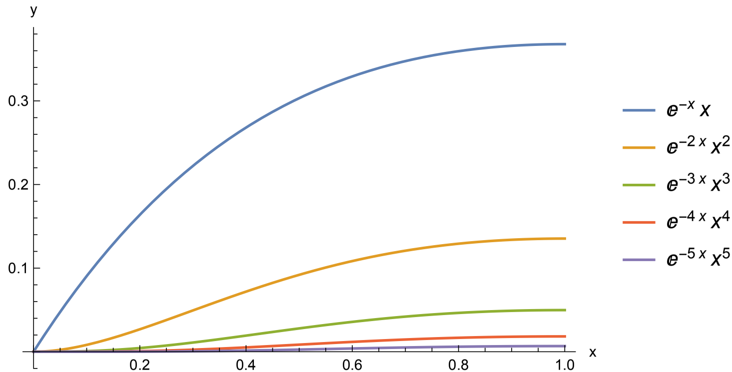
$$|f_n(x) - f(x)| < \epsilon_n$$

for all $n \in \mathbb{N}$ and for all $x \in D$.

Remark: an infinite series converges uniformly if its sequence of partial sums converges uniformly.

Illustration

Sequence $\{x^n e^{-nx}\}_{n=1}^{\infty}$ converges uniformly to $f(x) = 0$ on $D = [0, \infty)$.



Justification

- Let $f_n(x) = x^n e^{-nx}$ for $n \in \mathbb{N}$ and $x \geq 0$.
- By l'Hôpital's Rule $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x^n}{e^{nx}} = 0$ for all $x \geq 0$, thus $f_n(x) \to 0$ pointwise for all $x \geq 0$. Define $f(x) = 0$.
- The First Derivative Test shows that $0 \leq f_n(x) \leq f_n(1) = e^{-n}$ for all $x \geq 0$, so let $\epsilon_n = e^{-n}$ and note that $\lim\limits_{n\to\infty} \epsilon_n = 0$.
- The sequence of functions $f_n(x) \to f(x)$ uniformly for $x \geq 0$ since $|f_n(x) - f(0)| = f_n(x) \leq \epsilon_n$ for all $n \in \mathbb{N}$.

Examples (1 of 3)

Sequence $\{x^n\}_{n=1}^{\infty}$ converges pointwise on $[0, 1]$ to

$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$ but does not converge uniformly.

Justification

- Let $x \in [0, 1)$ then $\lim\limits_{n\to\infty} x^n = 0$. Also, $\lim_{n\to\infty} 1^n = 1$, so $x^n \to f(x)$ pointwise for $0 \le x \le 1$.
- Let $\{\epsilon_n\}_{n=1}^{\infty}$ be any sequence of positive real numbers such that $\lim\limits_{n\to\infty} \epsilon_n = 0$. Without loss of generality assume $0 < \epsilon_n < 1$ for all $n \in \mathbb{N}$.
- Note that $0 < 1 - \epsilon_n < 1$ for all $n \in \mathbb{N}$ and thus $0 < (1 - \epsilon_n)^{1/n} < 1$ as well. Therefore for all $n \in \mathbb{N}$ there exists x_n such that $(1 - \epsilon_n)^{1/n} < x_n < 1$.
- Consider $x_n^n > 1 - \epsilon_n > 0$ and thus $x_n^n \not\to 0$ as $n \to \infty$ and the convergence is not uniform.

Examples (2 of 3)

The sequence $\{x^n\}_{n=1}^{\infty}$ converges uniformly to $f(x) = 0$ on $[0, b]$ for any $0 < b < 1$.

Justification

- As in the previous example $x^n \to 0$ pointwise for all $0 \leq x \leq b < 1$.
- Define $\epsilon_n = b^n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \epsilon_n = 0$.
- Note that $x^n \leq b^n$ for all $0 \leq x \leq b$, or equivalently, $|x^n - 0| \leq \epsilon_n$ for all $0 \leq x \leq b$. Hence the convergence is uniform.

Examples (3 of 3)

The infinite series $\displaystyle\sum_{n=1}^{\infty} x^n$ converges pointwise on $(-1, 1)$ to $f(x) = \dfrac{x}{1-x}$ but not uniformly.

Justification

- The Nth partial sum of the series is

$$f_N(x) = x + x^2 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x} - 1.$$

- For any $-1 < x < 1$,

$$\lim_{N\to\infty} f_N(x) = \lim_{N\to\infty} \left[\frac{1 - x^{N+1}}{1 - x} - 1\right] = \frac{1}{1-x} - 1 = \frac{x}{1-x} = f(x),$$

(this proves pointwise convergence).

- Consider

$$\begin{aligned} \lim_{x\to 1^-} |f_N(x) - f(x)| &= \lim_{x\to 1^-} \left|\frac{1 - x^{N+1}}{1 - x} - 1 - \frac{x}{1-x}\right| \\ &= \lim_{x\to 1^-} \frac{x^{N+1}}{1-x} = \infty, \end{aligned}$$

so the convergence is not uniform.

Properties Preserved by Uniform Convergence

Suppose the series $\displaystyle\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to its sum $u(x)$ on $[a, b]$.

- If for each n, $u_n(x)$ is continuous on $[a, b]$, then the sum $u(x)$ is continuous on $[a, b]$.
- If for each n, $u_n(x)$ is integrable on $[a, b]$, then the sum $u(x)$ is integrable on $[a, b]$, and

$$\int_a^b u(x)\,dx = \int_a^b \left(\sum_{n=1}^{\infty} u_n(x) \right) dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx.$$

- If for each n, $u_n'(x)$ exists and $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on $[a, b]$, then the sum $u(x)$ is differentiable on $[a, b]$ and the derivative can be obtained by differentiating the series term by term,

$$u'(x) = \left(\sum_{n=1}^{\infty} u_n(x) \right)' = \sum_{n=1}^{\infty} u_n'(x) \quad \text{for all } x \in [a, b].$$

Weierstrass M-Test

The following theorem provides a convenient means of determining whether an infinite series converges uniformly.

Theorem

Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of functions defined on an interval $[a, b]$ and suppose that for each n there is a non-negative number M_n such that $|u_n(x)| \leq M_n$ for all $x \in [a, b]$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a, b]$.

Example

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$ converges uniformly on $(-\infty, \infty)$.

Since $\frac{1}{n^2}|\cos(nx)| \leq \frac{1}{n^2}$ for all $x \in \mathbb{R}$ and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, then the series converges uniformly for all $x \in \mathbb{R}$.

Homework

- Read Sections 3.6–3.7
- Exercises: 11–19