

# The Heat Equation

MATH 467 *Partial Differential Equations*

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# Objectives

In this lesson we will:

- ▶ solve the heat equation with Dirichlet boundary conditions,
- ▶ solve the heat equation with Neumann boundary conditions,
- ▶ solve the heat equation with Robin boundary conditions, and
- ▶ solve the heat equation with nonhomogeneous boundary conditions.

# Dirichlet Boundary Conditions

Find the solution to the IBVP:

$$u_t = k u_{xx} \quad \text{for } 0 < x < L, t > 0$$

$$u(0, t) = 0$$

$$u(L, t) = 0$$

$$u(x, 0) = \begin{cases} x & \text{for } 0 \leq x \leq L/2, \\ L - x & \text{for } L/2 < x \leq L. \end{cases}$$

## Solution (1 of 4)

The nontrivial product solutions must take the form,

$$u_n(x, t) = e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}$$

for  $n = 1, 2, \dots$ . By the Principle of Superposition,

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}$$

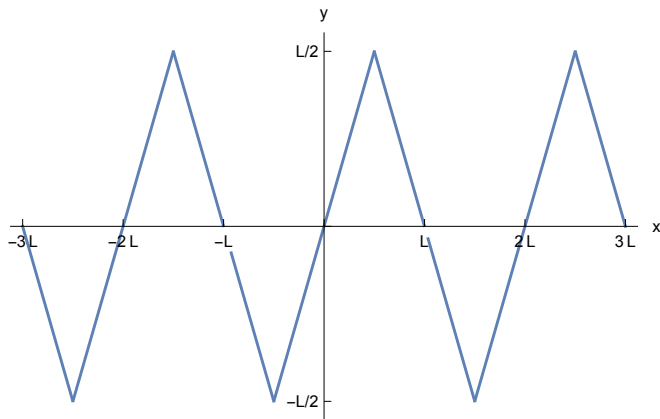
is also a solution to the PDE and satisfies the BCs. Applying the initial condition yields,

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = \begin{cases} x & \text{for } 0 \leq x \leq L/2, \\ L - x & \text{for } L/2 < x \leq L. \end{cases}$$

The coefficients  $b_n$  can be found using the Euler-Fourier integral formulas.

## Solution (2 of 4)

Think of  $f(x)$  as the  $2L$ -periodic, odd extension of the initial condition. In this case  $f(x)$  is continuous on  $(-\infty, \infty)$ .

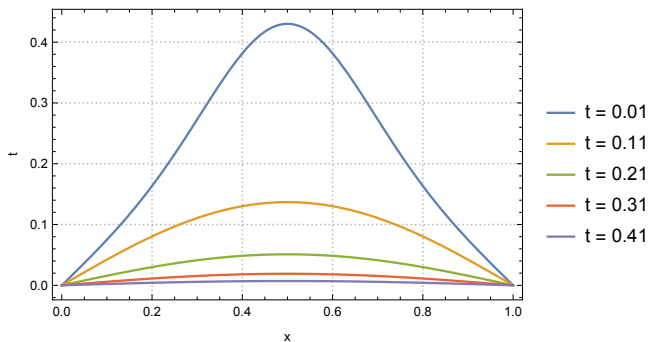


## Solution (3 of 4)

$$\begin{aligned}b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \\&= \frac{2}{L} \int_0^{L/2} x \sin \frac{n\pi x}{L} dx + \frac{2}{L} \int_{L/2}^L (L-x) \sin \frac{n\pi x}{L} dx \\&= \frac{L}{n^2\pi^2} \left( 2 \sin \frac{n\pi}{2} - n\pi \cos \frac{n\pi}{2} \right) + \frac{L}{n^2\pi^2} \left( n\pi \cos \frac{n\pi}{2} + 2 \sin \frac{n\pi}{2} \right) \\&= \frac{4L}{n^2\pi^2} \sin \frac{n\pi}{2}\end{aligned}$$

## Solution (4 of 4)

$$u(x, t) = \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2} e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}$$



$$k = 1 = L$$

# Neumann Boundary Conditions

Find the solution to the IBVP:

$$u_t = u_{xx} \quad \text{for } 0 < x < \pi, t > 0$$

$$u_x(0, t) = 0$$

$$u_x(\pi, t) = 0$$

$$u(x, 0) = \begin{cases} 0 & \text{for } 0 \leq x < \pi/2, \\ 100 & \text{for } \pi/2 \leq x \leq \pi. \end{cases}$$

## Solution (1 of 8)

Assuming a product solution of the form  $u(x, t) = X(x)T(t)$  then

$$\begin{aligned}X(x)T'(t) &= X''(x)T(t) \\ \frac{X(x)T'(t)}{X(x)T(t)} &= \frac{X''(x)T(t)}{X(x)T(t)} \\ \frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = c\end{aligned}$$

where  $c$  is a constant.

## Solution (2 of 8)

Suppose  $c = 0$ , then

$$\frac{X''(x)}{X(x)} = 0$$

$$X''(x) = 0$$

$$X(x) = Ax + B.$$

Since  $X'(0) = X'(\pi) = 0$  then  $A = 0$  and  $B = a_0/2$ , where  $a_0$  is an arbitrary constant.

## Solution (3 of 8)

Suppose  $c = -\lambda^2$  with  $\lambda > 0$ .

$$\frac{X''(x)}{X(x)} = -\lambda^2$$

$$X''(x) + \lambda^2 X(x) = 0$$

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x)$$

$$\text{and } X'(x) = -A\lambda \sin(\lambda x) + B\lambda \cos(\lambda x).$$

Since  $X'(0) = X'(\pi) = 0$  then  $B = 0$  and  $\lambda \equiv \lambda_n = n$  for  $n = 1, 2, \dots$  and  $A$  is an arbitrary constant. Thus

$$u_n(x, t) = e^{-n^2 t} \cos(nx)$$

satisfies the PDE and the BCs.

## Solution (4 of 8)

Suppose  $c = \lambda^2$  with  $\lambda > 0$ .

$$\frac{X''(x)}{X(x)} = \lambda^2$$

$$X''(x) - \lambda^2 X(x) = 0$$

$$X(x) = A \cosh(\lambda x) + B \sinh(\lambda x)$$

$$\text{and } X'(x) = A\lambda \sinh(\lambda x) + B\lambda \cosh(\lambda x).$$

Since  $X'(0) = X'(\pi) = 0$  then  $A = 0 = B$  and hence there are no nontrivial solutions in this case.

## Solution (5 of 8)

By the Principle of Superposition, the solution to the PDE satisfying the BCs can be written as

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos(nx).$$

Applying the initial condition yields,

$$u(x, 0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \begin{cases} 0 & \text{for } 0 \leq x < \pi/2, \\ 100 & \text{for } \pi/2 \leq x \leq \pi. \end{cases}$$

The coefficients  $a_n$  can be found using the Euler-Fourier integral formulas.

## Solution (6 of 8)

Think of  $f(x)$  as the  $2\pi$ -periodic, even extension of the initial condition.

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_{\pi/2}^{\pi} 100 dx = 100$$

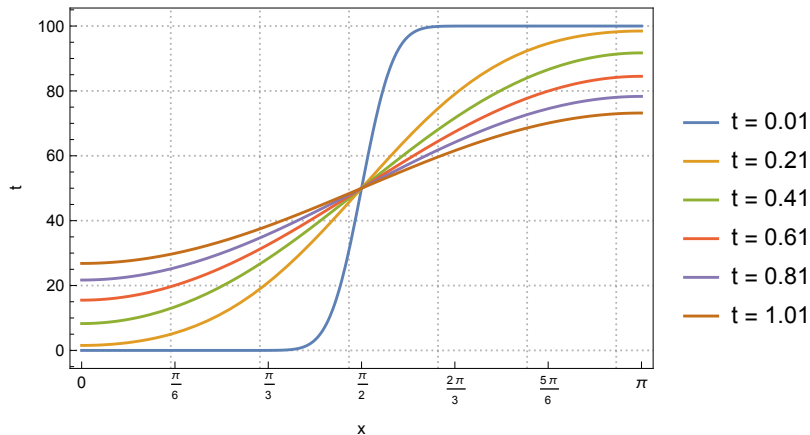
$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx \\ &= \frac{2}{\pi} \int_{\pi/2}^{\pi} 100 \cos(nx) dx \\ &= -\frac{200}{n\pi} \sin \frac{n\pi}{2} \end{aligned}$$

Consequently, the formal solution to the IBVP is

$$u(x, t) \sim 50 - \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} \cos(nx).$$

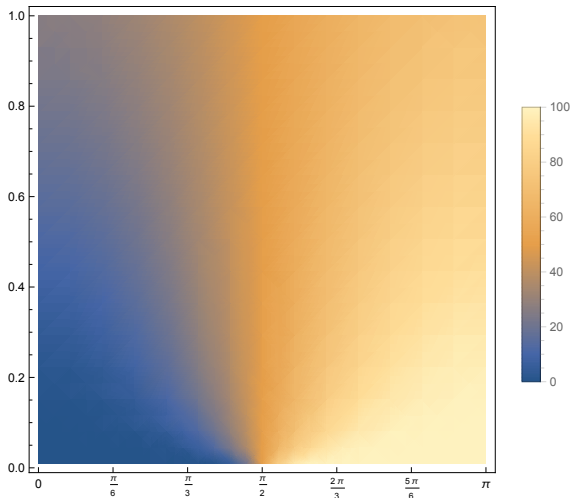
## Solution (7 of 8)

$$u(x, t) \sim 50 - \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} e^{-n^2 t} \cos(nx).$$



## Solution (8 of 8)

$$u(x, t) \sim 50 - \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} e^{-n^2 t} \cos(nx).$$



# Robin Boundary Conditions

Find the solution to the IBVP:

$$u_t = u_{xx} \quad \text{for } 0 < x < L, t > 0$$

$$u_x(0, t) = \alpha u(0, t)$$

$$u_x(L, t) = -\beta u(L, t)$$

$$u(x, 0) = \begin{cases} 0 & \text{for } 0 \leq x < L/2, \\ 100 & \text{for } L/2 \leq x \leq L. \end{cases}$$

Assume  $\alpha > 0$  and  $\beta > 0$ .

## Solution (1 of 10)

Assuming a product solution of the form  $u(x, t) = X(x)T(t)$  then

$$\begin{aligned}X(x)T'(t) &= X''(x)T(t) \\ \frac{X(x)T'(t)}{X(x)T(t)} &= \frac{X''(x)T(t)}{X(x)T(t)} \\ \frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = c\end{aligned}$$

where  $c$  is a constant.

## Solution (2 of 10)

Suppose  $c = 0$ , then

$$\frac{X''(x)}{X(x)} = 0$$

$$X''(x) = 0$$

$$X(x) = Ax + B.$$

**Question:** can such a function satisfy the boundary conditions:

$$A = \alpha B \quad (\text{BC at } x = 0)$$

$$A = -\beta(AL + B) \quad (\text{BC at } x = L)$$

Eliminating  $A$  from the 2nd equation using the 1st equation yields:

$$\alpha B = -\beta(\alpha L + 1)B.$$

If  $B = 0$  then  $A = 0$  and  $X(x) = 0$ . If  $B \neq 0$  then

$$\alpha = -\beta(\alpha L + 1) \implies \alpha < 0 \quad (\text{contradiction}).$$

## Solution (3 of 10)

Suppose  $c = \lambda^2$  with  $\lambda > 0$ .

$$\frac{X''(x)}{X(x)} = \lambda^2$$

$$X''(x) - \lambda^2 X(x) = 0$$

$$X(x) = A \cosh(\lambda x) + B \sinh(\lambda x)$$

$$\text{and } X'(x) = A\lambda \sinh(\lambda x) + B\lambda \cosh(\lambda x).$$

Can this type of function satisfy the boundary conditions?

$$B\lambda = \alpha A$$

$$A\lambda \sinh(\lambda L) + B\lambda \cosh(\lambda L) = -\beta(A \cosh(\lambda L) + B \sinh(\lambda L))$$

The first equation implies  $A = 0 \iff B = 0$ .

## Solution (4 of 10)

Use the BC at  $x = 0$  to eliminate  $A$  and  $B$  from the second equation,

$$A\lambda \sinh(\lambda L) + \frac{\alpha A}{\lambda} \lambda \cosh(\lambda L) = -\beta(A \cosh(\lambda L) + \frac{\alpha A}{\lambda} \sinh(\lambda L))$$

$$\lambda \sinh(\lambda L) + \alpha \cosh(\lambda L) = -\beta(\cosh(\lambda L) + \frac{\alpha}{\lambda} \sinh(\lambda L))$$

$$\left(\lambda + \frac{\alpha\beta}{\lambda}\right) \sinh(\lambda L) = -(\alpha + \beta) \cosh(\lambda L)$$

$$\left(\lambda + \frac{\alpha\beta}{\lambda}\right) \tanh(\lambda L) = -(\alpha + \beta)$$

since  $\cosh(\lambda L) \geq 1$ .

## Solution (5 of 10)

$$\left(\lambda + \frac{\alpha}{\lambda}\right) \tanh(\lambda L) = -(\alpha + \beta)$$

Since  $\alpha > 0$ ,  $\beta > 0$ ,  $L > 0$ , and we have assumed  $\lambda > 0$  then

$$\tanh(\lambda L) < 0 \iff \lambda L < 0$$

which is a contradiction. Hence there are no nontrivial solutions when  $c > 0$ .

## Solution (6 of 10)

Suppose  $c = -\lambda^2$  with  $\lambda > 0$ .

$$\frac{X''(x)}{X(x)} = -\lambda^2$$

$$X''(x) + \lambda^2 X(x) = 0$$

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x)$$

$$\text{and } X'(x) = -A\lambda \sin(\lambda x) + B\lambda \cos(\lambda x).$$

Can this type of function satisfy the boundary conditions?

$$B\lambda = \alpha A$$

$$-A\lambda \sin(\lambda L) + B\lambda \cos(\lambda L) = -\beta(A \cos(\lambda L) + B \sin(\lambda L))$$

The first equation implies  $A = 0 \iff B = 0$ .

## Solution (7 of 10)

Use the BC at  $x = 0$  to eliminate  $A$  and  $B$  from the second equation,

$$-A\lambda \sin(\lambda L) + \frac{\alpha A}{\lambda} \lambda \cos(\lambda L) = -\beta(A \cos(\lambda L) + \frac{\alpha A}{\lambda} \sin(\lambda L))$$

$$-\lambda \sin(\lambda L) + \alpha \cos(\lambda L) = -\beta(\cos(\lambda L) + \frac{\alpha}{\lambda} \sin(\lambda L))$$

$$\left(\lambda - \frac{\alpha\beta}{\lambda}\right) \sin(\lambda L) = (\alpha + \beta) \cos(\lambda L)$$

**Note:**

$$\cos(\lambda L) = 0 \iff \lambda = \frac{(2n-1)\pi}{2L}$$

with  $n \in \mathbb{N}$ .

## Solution (8 of 10)

Suppose  $\lambda = \frac{(2n-1)\pi}{2L}$  with  $n \in \mathbb{N}$ , then  $\sin(\lambda L) = \pm 1$  and

$$\left(\lambda - \frac{\alpha\beta}{\lambda}\right) \sin(\lambda L) = (\alpha + \beta) \cos(\lambda L)$$

$$\lambda - \frac{\alpha\beta}{\lambda} = 0$$

$$\alpha\beta = \left(\frac{(2n-1)\pi}{2L}\right)^2.$$

If there is no  $n \in \mathbb{N}$  satisfying the last equation, then  $\cos(\lambda L) \neq 0$  and

$$\left(\lambda - \frac{\alpha\beta}{\lambda}\right) \sin(\lambda L) = (\alpha + \beta) \cos(\lambda L)$$

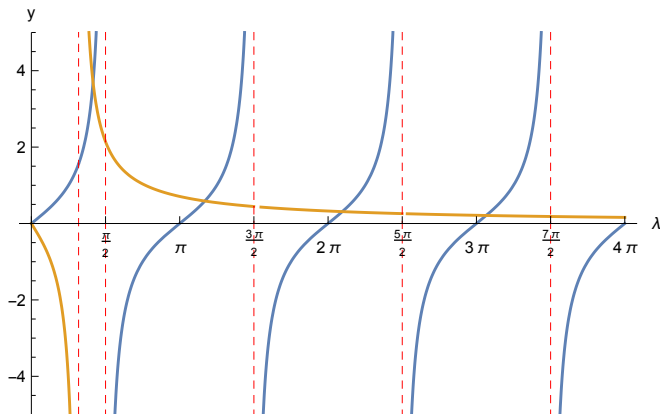
$$\left(\lambda - \frac{\alpha\beta}{\lambda}\right) \tan(\lambda L) = \alpha + \beta$$

$$\tan(\lambda L) = \frac{\alpha + \beta}{\lambda - \frac{\alpha\beta}{\lambda}} = \frac{(\alpha + \beta)\lambda}{\lambda^2 - \alpha\beta}.$$

## Solution (9 of 10)

$$\tan(\lambda L) = \frac{(\alpha + \beta)\lambda}{\lambda^2 - \alpha\beta}$$

This equation has positive solutions which can be approximated using Newton's method.



$$L = \alpha = \beta = 1$$

## Solution (10 of 10)

The first few solutions are

$$\lambda_1 \approx 1.20779$$

$$\lambda_2 \approx 3.44824$$

$$\lambda_3 \approx 6.44095$$

$$\lambda_4 \approx 9.53048$$

and  $\lambda_n \rightarrow (n-1)\pi$  as  $n \rightarrow \infty$ .

The corresponding eigenfunctions are

$$X_n(x) = A_n \cos(\lambda_n x) + B_n \sin(\lambda_n x)$$

while the product solutions have the form

$$u_n(x, t) = e^{-\lambda_n^2 t} (A_n \cos(\lambda_n x) + B_n \sin(\lambda_n x)).$$

# Nonhomogeneous Boundary Conditions

Consider the IBVP:

$$u_t = ku_{xx} \quad \text{for } 0 < x < L, t > 0$$

$$u(0, t) = u_0$$

$$u(L, t) = u_L$$

$$u(x, 0) = f(x)$$

Recall that the temperature in the region asymptotically approached a steady state of 0 when the boundary conditions were homogeneous ( $u(0, t) = u(L, t) = 0$ ). Suppose that in the nonhomogeneous case the temperature distribution approaches a nonzero steady state  $U(x)$ .

# Steady-State Solution

Suppose

$$U''(x) = 0 \quad \text{for } 0 < x < L$$

$$U(0) = u_0$$

$$U(L) = u_L$$

then  $U(x) = Ax + B$ .

$$U(0) = u_0 = A(0) + B \implies B = u_0$$

The other boundary condition implies

$$U(L) = u_L = AL + u_0 \implies A = \frac{u_L - u_0}{L}.$$

Hence the steady-state solution is

$$U(x) = \frac{(u_L - u_0)x}{L} + u_0.$$

## Time-Dependent Solution (1 of 2)

Assume the solution to the IBVP can be written as

$u(x, t) = U(x) + v(x, t)$  where  $v(x, t)$  is an unknown function.

**Question:** what IBVP must  $v(x, t)$  satisfy?

$$u_t = ku_{xx}$$

$$(U(x) + v(x, t))_t = k(U(x) + v(x, t))_{xx}$$

$$v_t = kv_{xx} \quad (\text{PDE})$$

$$u(0, t) = u_0$$

$$U(0) + v(0, t) = u_0$$

$$v(0, t) = 0 \quad (\text{BC at } x = 0)$$

Likewise  $v(L, t) = 0$ .

$$u(x, 0) = f(x)$$

$$U(x) + v(x, 0) = f(x)$$

$$v(x, 0) = f(x) - U(x) \quad (\text{IC at } t = 0)$$

## Time-Dependent Solution (2 of 2)

$$v_t = kv_{xx} \quad \text{for } 0 < x < L \text{ and } t > 0$$

$$v(0, t) = 0$$

$$v(L, t) = 0$$

$$v(x, 0) = f(x) - U(x)$$

$$\text{If } b_n = \frac{2}{L} \int_0^L (f(x) - U(x)) \sin \frac{n\pi x}{L} dx,$$

$$\text{then } v(x, t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L},$$

$$\text{and } u(x, t) = u_0 + \frac{(u_L - u_0)x}{L} + \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}.$$

## Example

Solve the following IBVP:

$$u_t = 5u_{xx} \quad \text{for } 0 < x < 1 \text{ and } t > 0$$

$$u(0, t) = 10$$

$$u(1, t) = 20$$

$$u(x, 0) = 10 + 11x - x^2$$

## Solution (1 of 2)

The steady-state solution is

$$U(x) = 10 + \frac{(20 - 10)x}{1} = 10(1 + x).$$

The transient solution  $v(x, t)$  satisfies the IBVP:

$$v_t = 5v_{xx} \quad \text{for } 0 < x < 1 \text{ and } t > 0$$

$$v(0, t) = 0$$

$$v(1, t) = 0$$

$$v(x, 0) = 10 + 11x - x^2 - 10(1 + x) = x - x^2$$

## Solution (2 of 2)

$$\begin{aligned} b_n &= 2 \int_0^1 (x - x^2) \sin(n\pi x) dx \\ &= \frac{4(1 - (-1)^n)}{n^3 \pi^3} \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ 8/(n^3 \pi^3) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

$$v(x, t) = \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{e^{-5(2n-1)^2 \pi^2 t}}{(2n-1)^3} \sin((2n-1)\pi x)$$

$$u(x, t) = 10(1 + x) + \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{e^{-5(2n-1)^2 \pi^2 t}}{(2n-1)^3} \sin((2n-1)\pi x)$$

# Time Dependent Boundary Conditions

Consider the IBVP:

$$u_t = ku_{xx}, \quad \text{for } 0 < x < L \text{ and } t > 0$$

$$u(0, t) = a(t)$$

$$u(L, t) = b(t)$$

$$u(x, 0) = f(x)$$

- ▶ In this situation the boundary conditions are functions of time.
- ▶ There may not be a steady-state solution, but the approach used in the case of constant, nonhomogeneous BCs is useful.

Define a **reference function**  $r(x, t) = a(t) + \frac{(b(t) - a(t))x}{L}$  and suppose  $u(x, t) = r(x, t) + v(x, t)$ .

**Question:** what IBVP does  $v(x, t)$  satisfy?

## PDE for $v(x, t)$

$$u_t = ku_{xx}$$

$$(r(x, t) + v(x, t))_t = k(r(x, t) + v(x, t))_{xx}$$

$$a'(t) + \frac{(b'(t) - a'(t))x}{L} + v_t = kv_{xx}$$

$$v_t = kv_{xx} - a'(t) - \frac{(b'(t) - a'(t))x}{L}$$

**Note:** this PDE is nonhomogeneous.

## BCs for $v(x, t)$

$$u(0, t) = a(t)$$

$$r(0, t) + v(0, t) = a(t)$$

$$a(t) + v(0, t) = a(t)$$

$$v(0, t) = 0 \quad (\text{BC at } x = 0)$$

$$u(L, t) = b(t)$$

$$r(L, t) + v(L, t) = b(t)$$

$$b(t) + v(L, t) = b(t)$$

$$v(L, t) = 0 \quad (\text{BC at } x = L)$$

The boundary conditions are once again homogeneous.

## IC for $v(x, t)$

$$u(x, 0) = f(x)$$

$$r(x, 0) + v(x, 0) = f(x)$$

$$a(0) + \frac{(b(0) - a(0))x}{L} + v(x, 0) = f(x)$$

$$v(x, 0) = f(x) - a(0) - \frac{(b(0) - a(0))x}{L}$$

## IBVP for $v(x, t)$

In summary:

$$v_t = kv_{xx} - a'(t) - \frac{(b'(t) - a'(t))x}{L} \quad \text{for } 0 < x < L \text{ and } t > 0$$

$$v(0, t) = 0$$

$$v(L, t) = 0$$

$$v(x, 0) = f(x) - a(0) - \frac{(b(0) - a(0))x}{L}$$

### Remarks:

- ▶ This IBVP has a nonhomogeneous PDE and a non-trivial initial condition.
- ▶ Search for a solution  $v(x, t) = v_1(x, t) + v_2(x, t)$  for which  $v_1(x, t)$  solves the homogeneous PDE with a non-zero initial condition and  $v_2(x, t)$  solves a nonhomogeneous PDE with a zero initial condition.

## Sum of Solutions (1 of 2)

Consider the two IBVPs with  $0 < x < L$  and  $t > 0$ :

$$(v_1)_t = k(v_1)_{xx}$$

$$v_1(0, t) = 0$$

$$v_1(L, t) = 0$$

$$v_1(x, 0) = f(x) - \frac{(b(0) - a(0))x}{L} - a(0)$$

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$$(v_2)_t = k(v_2)_{xx} - a'(t) - \frac{(b'(t) - a'(t))x}{L}$$

$$v_2(0, t) = 0$$

$$v_2(L, t) = 0$$

$$v_2(x, 0) = 0$$

## Sum of Solutions (2 of 2)

If  $v_1(x, t)$  solves the 1st IBVP and  $v_2(x, t)$  solves the 2nd IBVP, then  $v(x, t) = v_1(x, t) + v_2(x, t)$  solves:

$$v_t = kv_{xx} - a'(t) - \frac{(b'(t) - a'(t))x}{L} \quad \text{for } 0 < x < L \text{ and } t > 0$$

$$v(0, t) = 0$$

$$v(L, t) = 0$$

$$v(x, 0) = f(x) - a(0) - \frac{(b(0) - a(0))x}{L}$$

# Solution of IBVP with Homogeneous PDE

The IBVP,

$$\begin{aligned}(v_1)_t &= k(v_1)_{xx} \text{ for } 0 < x < L, t > 0 \\ v_1(0, t) &= 0 \\ v_1(L, t) &= 0 \\ v_1(x, 0) &= f(x) - \frac{(b(0) - a(0))x}{L} - a(0)\end{aligned}$$

has solution

$$v_1(x, t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L} \int_0^L \left( f(x) - \frac{(b(0) - a(0))x}{L} - a(0) \right) \sin \frac{n\pi x}{L} dx.$$

# Solution of IBVP with Nonhomogeneous PDE

We will assume the solution to the IBVP:

$$(v_2)_t = k(v_2)_{xx} - a'(t) - \frac{(b'(t) - a'(t))x}{L} \text{ for } 0 < x < L, t > 0$$

$$v_2(0, t) = 0$$

$$v_2(L, t) = 0$$

$$v_2(x, 0) = 0$$

is of the form

$$v_2(x, t) = \sum_{n=1}^{\infty} b_n(t) e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}$$

where the coefficients  $b_n$  are functions of  $t$ . Differentiate this solution and substitute it into the nonhomogeneous PDE.

## Verifying the Solution

$$(v_2)_{xx} = - \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{L^2} b_n(t) e^{-kn^2 \pi^2 t / L^2} \sin \frac{n\pi x}{L}$$

$$(v_2)_t = \sum_{n=1}^{\infty} \left( b'_n(t) e^{-kn^2 \pi^2 t / L^2} - \frac{kn^2 \pi^2}{L^2} b_n(t) e^{-kn^2 \pi^2 t / L^2} \right) \sin \frac{n\pi x}{L}$$

Since  $(v_2)_t - k(v_2)_{xx} = -a'(t) - \frac{(b'(t) - a'(t))x}{L}$  then

$$\sum_{n=1}^{\infty} b'_n(t) e^{-kn^2 \pi^2 t / L^2} \sin \frac{n\pi x}{L} = -a'(t) - \frac{(b'(t) - a'(t))x}{L}.$$

Multiply both sides by  $\sin(m\pi x / L)$  and integrate over  $[0, L]$ .

# Integration and Orthogonality

$$\begin{aligned}\frac{L}{2}b'_m(t)e^{-km^2\pi^2t/L^2} &= -\int_0^L \left( a'(t) + \frac{(b'(t) - a'(t))x}{L} \right) \sin \frac{m\pi x}{L} dx \\ &= -\frac{L}{m\pi} (a'(t) - (-1)^m b'(t)) \\ b'_m(t) &= \frac{2}{m\pi} e^{km^2\pi^2t/L^2} ((-1)^m b'(t) - a'(t))\end{aligned}$$

Integrate both sides with respect to  $t$ .

$$\begin{aligned}\int_0^t b'_m(s) ds &= \frac{2}{m\pi} \int_0^t e^{km^2\pi^2s/L^2} ((-1)^m b'(s) - a'(s)) ds \\ b_m(t) - b_m(0) &= \frac{2}{m\pi} \int_0^t e^{km^2\pi^2s/L^2} ((-1)^m b'(s) - a'(s)) ds\end{aligned}$$

Since  $v_2(x, 0) = 0$  then  $b_m(0) = 0$  for  $m \in \mathbb{N}$ .

# Assembling the Solution

$$r(x, t) = a(t) + \frac{(b(t) - a(t))x}{L}$$

$$v_1(x, t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}$$

$$v_2(x, t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[ \int_0^t e^{-kn^2\pi^2(t-s)/L^2} ((-1)^n b'(s) - a'(s)) ds \right] \sin \frac{n\pi x}{L}$$

$$u(x, t) = r(x, t) + v_1(x, t) + v_2(x, t)$$

## Example

Find the solution to the following IBVP.

$$u_t = u_{xx} \text{ for } 0 < x < 1 \text{ and } t > 0$$

$$u(0, t) = t$$

$$u(1, t) = 1 + \sin t$$

$$u(x, 0) = x$$

The reference function is

$$r(x, t) = t + (1 - t + \sin t)x.$$

## Associated Homogeneous IVBP

$$(v_1)_t = (v_1)_{xx} \quad \text{for } 0 < x < 1 \text{ and } t > 0$$

$$v_1(0, t) = 0$$

$$v_1(1, t) = 0$$

$$v_1(x, 0) = x - (0 + (1 - 0 + \sin 0)x) = x - x = 0$$

Hence  $v_1(x, t) = 0$ .

# Associated Nonhomogeneous IVBP

$$(v_2)_t = (v_2)_{xx} - 1 + (1 - \cos t)x \quad \text{for } 0 < x < 1 \text{ and } t > 0$$

$$v_2(0, t) = 0$$

$$v_2(1, t) = 0$$

$$v_2(x, 0) = 0$$

Assume solution  $v_2(x, t)$  has the form

$$v_2(x, t) = \sum_{n=1}^{\infty} b_n(t) e^{-n^2 \pi^2 t} \sin(n\pi x)$$

where

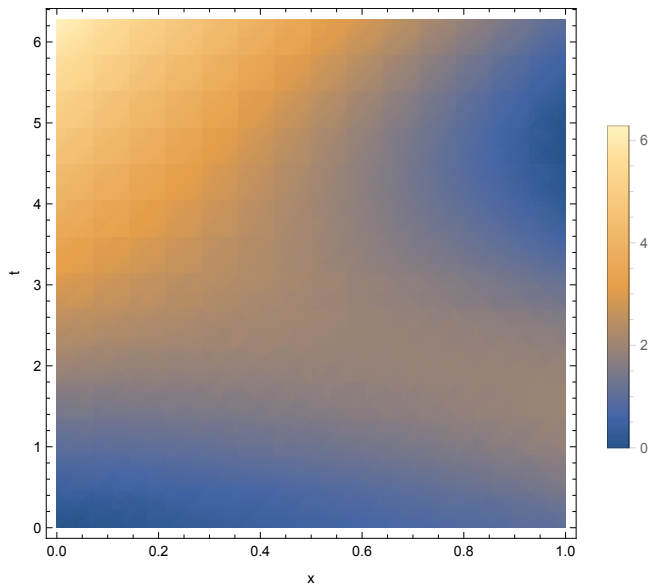
$$b_n(t) = \frac{2}{n\pi} \int_0^t e^{n^2 \pi^2 s} ((-1)^n \cos s - 1) ds.$$

## Calculation of $b_n(t)$

$$\begin{aligned} b_n(t) &= \frac{2}{n\pi} \int_0^t e^{n^2\pi^2 s} ((-1)^n \cos s - 1) ds \\ &= \frac{1 - e^{n^2\pi^2 t}}{n^2\pi^2} - \frac{(-1)^n n^2\pi^2}{1 + n^4\pi^4} + \frac{(-1)^n e^{n^2\pi^2 t}}{1 + n^4\pi^4} (n^2\pi^2 \cos t + \sin t) \end{aligned}$$

$$u(x, t) = t + (1 - t + \sin t)x + \sum_{n=1}^{\infty} b_n(t) e^{-n^2\pi^2 t} \sin(n\pi x)$$

# Graph



# Homework

- ▶ Read Section 4.2
- ▶ Exercises: 4, 5, 6