The Heat Equation

MATH 467 *Partial Differential Equations*

J Robert Buchanan

Department of Mathematics

Fall 2022

Objectives

In this lesson we will:

- solve the heat equation with Dirichlet boundary conditions,
- solve the heat equation with Neumann boundary conditions,
- solve the heat equation with Robin boundary conditions, and
- solve the heat equation with nonhomogeneous boundary conditions.

Dirichlet Boundary Conditions

Find the solution to the IBVP:

$$\begin{aligned} u_t &= k\, u_{xx} \quad \text{for } 0 < x < L,\ t > 0 \\ u(0,t) &= 0 \\ u(L,t) &= 0 \\ u(x,0) &= \begin{cases} x & \text{for } 0 \le x \le L/2, \\ L - x & \text{for } L/2 < x \le L. \end{cases} \end{aligned}$$

Solution (1 of 4)

The nontrivial product solutions must take the form,

$$u_n(x,t) = e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}$$

for $n = 1, 2, \ldots$. By the Principle of Superposition,

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}$$

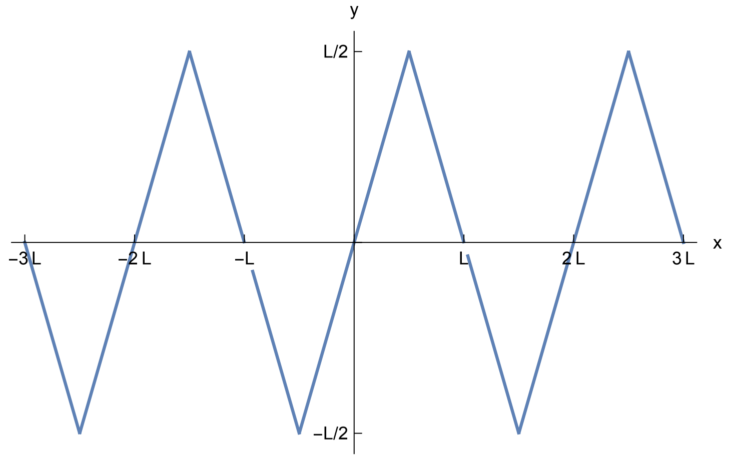
is also a solution to the PDE and satisfies the BCs. Applying the initial condition yields,

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = \begin{cases} x & \text{for } 0 \le x \le L/2, \\ L - x & \text{for } L/2 < x \le L. \end{cases}$$

The coefficients b_n can be found using the Euler-Fourier integral formulas.

Solution (2 of 4)

Think of $f(x)$ as the $2L$-periodic, odd extension of the initial condition. In this case $f(x)$ is continuous on $(-\infty, \infty)$.

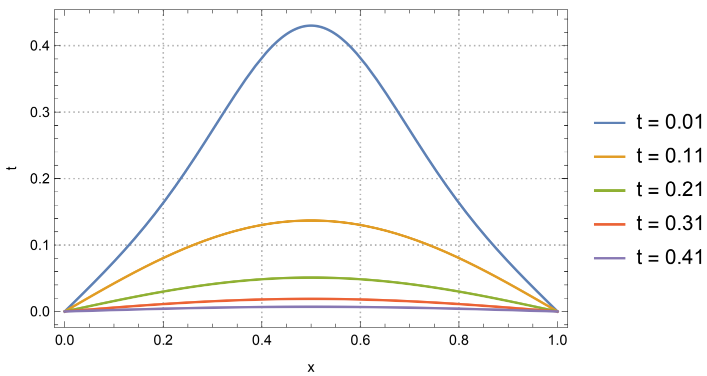


Solution (3 of 4)

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \\
&= \frac{2}{L}\int_0^{L/2} x\sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}\,dx \\
&= \frac{L}{n^2\pi^2}\left(2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\right) + \frac{L}{n^2\pi^2}\left(n\pi\cos\frac{n\pi}{2} + 2\sin\frac{n\pi}{2}\right) \\
&= \frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}
\end{aligned}
$$

Solution (4 of 4)

$$u(x,t) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}e^{-kn^2\pi^2 t/L^2}\sin\frac{n\pi x}{L}$$



$$k = 1 = L$$

Neumann Boundary Conditions

Find the solution to the IBVP:

$$
\begin{aligned}
u_t &= u_{xx} \quad \text{for } 0 < x < \pi,\ t > 0 \\
u_x(0,t) &= 0 \\
u_x(\pi,t) &= 0 \\
u(x,0) &= \begin{cases} 0 & \text{for } 0 \le x < \pi/2, \\ 100 & \text{for } \pi/2 \le x \le \pi. \end{cases}
\end{aligned}
$$

Solution (1 of 8)

Assuming a product solution of the form $u(x,t) = X(x)T(t)$ then

$$\begin{aligned} X(x)T'(t) &= X''(x)T(t) \\ \frac{X(x)T'(t)}{X(x)T(t)} &= \frac{X''(x)T(t)}{X(x)T(t)} \\ \frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = c \end{aligned}$$

where c is a constant.

Solution (2 of 8)

Suppose $c = 0$, then

$$\begin{aligned} \frac{X''(x)}{X(x)} &= 0 \\ X''(x) &= 0 \\ X(x) &= Ax + B. \end{aligned}$$

Since $X'(0) = X'(\pi) = 0$ then $A = 0$ and $B = a_0/2$, where a_0 is an arbitrary constant.

Solution (3 of 8)

Suppose $c = -\lambda^2$ with $\lambda > 0$.

$$\begin{aligned} \frac{X''(x)}{X(x)} &= -\lambda^2 \\ X''(x) + \lambda^2 X(x) &= 0 \\ X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \\ \text{and } X'(x) &= -A\lambda\sin(\lambda x) + B\lambda\cos(\lambda x). \end{aligned}$$

Since $X'(0) = X'(\pi) = 0$ then $B = 0$ and $\lambda \equiv \lambda_n = n$ for $n = 1, 2, \ldots$ and A is an arbitrary constant. Thus

$$u_n(x,t) = e^{-n^2 t}\cos(nx)$$

satisfies the PDE and the BCs.

Solution (4 of 8)

Suppose $c = \lambda^2$ with $\lambda > 0$.

$$\begin{aligned} \frac{X''(x)}{X(x)} &= \lambda^2 \\ X''(x) - \lambda^2 X(x) &= 0 \\ X(x) &= A\cosh(\lambda x) + B\sinh(\lambda x) \\ \text{and } X'(x) &= A\lambda\sinh(\lambda x) + B\lambda\cosh(\lambda x). \end{aligned}$$

Since $X'(0) = X'(\pi) = 0$ then $A = 0 = B$ and hence there are no nontrivial solutions in this case.

Solution (5 of 8)

By the Principle of Superposition, the solution to the PDE satisfying the BCs can be written as

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos(nx).$$

Applying the initial condition yields,

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \begin{cases} 0 & \text{for } 0 \le x < \pi/2, \\ 100 & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

The coefficients a_n can be found using the Euler-Fourier integral formulas.

Solution (6 of 8)

Think of $f(x)$ as the 2π-periodic, even extension of the initial condition.

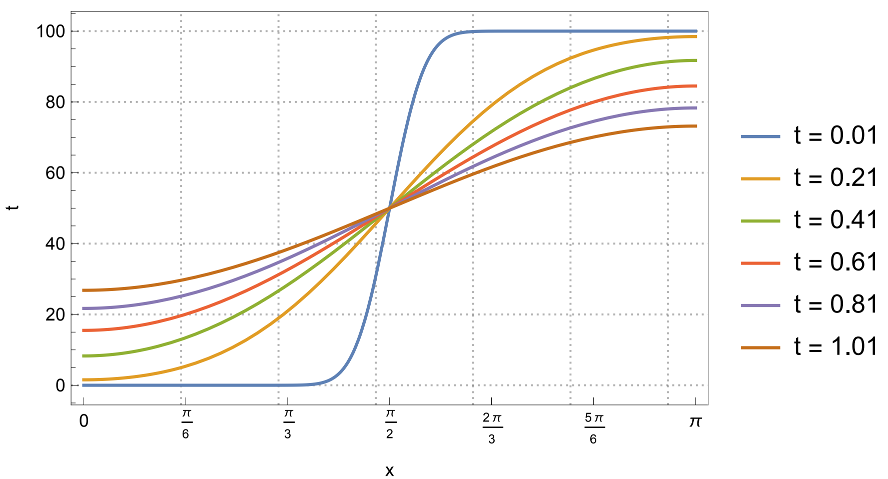
$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_{\pi/2}^{\pi} 100\,dx = 100 \\ a_n &= \frac{2}{\pi} \int_0^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{2}{\pi} \int_{\pi/2}^{\pi} 100\cos(nx)\,dx \\ &= -\frac{200}{n\pi} \sin\frac{n\pi}{2} \end{aligned}$$

Consequently, the formal solution to the IBVP is

$$u(x,t) \sim 50 - \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n} \cos(nx).$$

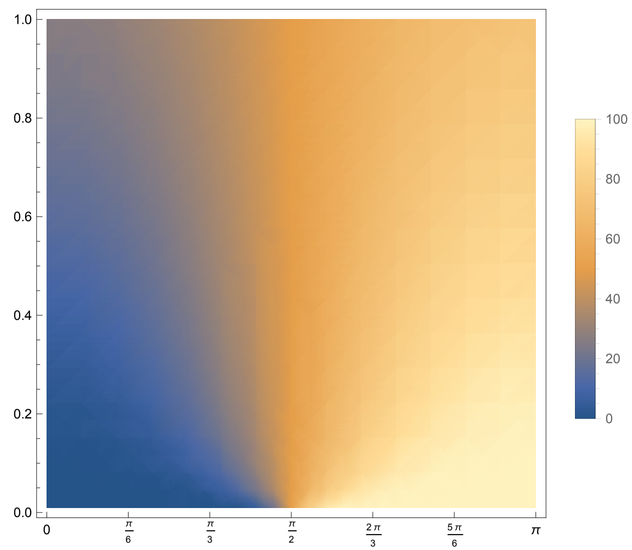
Solution (7 of 8)

$$u(x,t) \sim 50 - \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n} e^{-n^2 t} \cos(nx).$$



Solution (8 of 8)

$$u(x,t) \sim 50 - \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n} e^{-n^2 t} \cos(nx).$$



Robin Boundary Conditions

Find the solution to the IBVP:

$$\begin{aligned} u_t &= u_{xx} \quad \text{for } 0 < x < L,\ t > 0 \\ u_x(0,t) &= \alpha\, u(0,t) \\ u_x(L,t) &= -\beta\, u(L,t) \\ u(x,0) &= \begin{cases} 0 & \text{for } 0 \le x < L/2, \\ 100 & \text{for } L/2 \le x \le L. \end{cases} \end{aligned}$$

Assume $\alpha > 0$ and $\beta > 0$.

Solution (1 of 10)

Assuming a product solution of the form $u(x,t) = X(x)T(t)$ then

$$\begin{aligned} X(x)T'(t) &= X''(x)T(t) \\ \frac{X(x)T'(t)}{X(x)T(t)} &= \frac{X''(x)T(t)}{X(x)T(t)} \\ \frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = c \end{aligned}$$

where c is a constant.

Solution (2 of 10)

Suppose $c = 0$, then

$$\begin{aligned} \frac{X''(x)}{X(x)} &= 0 \\ X''(x) &= 0 \\ X(x) &= Ax + B. \end{aligned}$$

Question: can such a function satisfy the boundary conditions:

$$\begin{aligned} A &= \alpha B \quad (\text{BC at } x = 0) \\ A &= -\beta(AL + B) \quad (\text{BC at } x = L) \end{aligned}$$

Eliminating A from the 2nd equation using the 1st equation yields:

$$\alpha B = -\beta(\alpha L + 1)B.$$

If $B = 0$ then $A = 0$ and $X(x) = 0$. If $B \neq 0$ then

$$\alpha = -\beta(\alpha L + 1) \implies \alpha < 0 \quad (\text{contradiction}).$$

Solution (3 of 10)

Suppose $c = \lambda^2$ with $\lambda > 0$.

$$\begin{aligned} \frac{X''(x)}{X(x)} &= \lambda^2 \\ X''(x) - \lambda^2 X(x) &= 0 \\ X(x) &= A\cosh(\lambda x) + B\sinh(\lambda x) \\ \text{and } X'(x) &= A\lambda\sinh(\lambda x) + B\lambda\cosh(\lambda x). \end{aligned}$$

Can this type of function satisfy the boundary conditions?

$$\begin{aligned} B\lambda &= \alpha A \\ A\lambda\sinh(\lambda L) + B\lambda\cosh(\lambda L) &= -\beta(A\cosh(\lambda L) + B\sinh(\lambda L)) \end{aligned}$$

The first equation implies $A = 0 \iff B = 0$.

Solution (4 of 10)

Use the BC at $x = 0$ to eliminate A and B from the second equation,

$$
\begin{aligned}
A\lambda \sinh(\lambda L) + \frac{\alpha A}{\lambda}\lambda \cosh(\lambda L) &= -\beta\left(A\cosh(\lambda L) + \frac{\alpha A}{\lambda}\sinh(\lambda L)\right) \\
\lambda \sinh(\lambda L) + \alpha \cosh(\lambda L) &= -\beta\left(\cosh(\lambda L) + \frac{\alpha}{\lambda}\sinh(\lambda L)\right) \\
\left(\lambda + \frac{\alpha\beta}{\lambda}\right)\sinh(\lambda L) &= -(\alpha+\beta)\cosh(\lambda L) \\
\left(\lambda + \frac{\alpha\beta}{\lambda}\right)\tanh(\lambda L) &= -(\alpha+\beta)
\end{aligned}
$$

since $\cosh(\lambda L) \geq 1$.

Solution (5 of 10)

$$\left(\lambda + \frac{\alpha}{\lambda}\right)\tanh(\lambda L) = -(\alpha+\beta)$$

Since $\alpha > 0$, $\beta > 0$, $L > 0$, and we have assumed $\lambda > 0$ then

$$\tanh(\lambda L) < 0 \iff \lambda L < 0$$

which is a contradiction. Hence there are no nontrivial solutions when $c > 0$.

Solution (6 of 10)

Suppose $c = -\lambda^2$ with $\lambda > 0$.

$$
\begin{aligned}
\frac{X''(x)}{X(x)} &= -\lambda^2 \\
X''(x) + \lambda^2 X(x) &= 0 \\
X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \\
\text{and } X'(x) &= -A\lambda\sin(\lambda x) + B\lambda\cos(\lambda x).
\end{aligned}
$$

Can this type of function satisfy the boundary conditions?

$$
\begin{aligned}
B\lambda &= \alpha A \\
-A\lambda\sin(\lambda L) + B\lambda\cos(\lambda L) &= -\beta(A\cos(\lambda L) + B\sin(\lambda L))
\end{aligned}
$$

The first equation implies $A = 0 \iff B = 0$.

Solution (7 of 10)

Use the BC at $x = 0$ to eliminate A and B from the second equation,

$$
\begin{aligned}
-A\lambda \sin(\lambda L) + \frac{\alpha A}{\lambda}\lambda \cos(\lambda L) &= -\beta\left(A\cos(\lambda L) + \frac{\alpha A}{\lambda}\sin(\lambda L)\right) \\
-\lambda \sin(\lambda L) + \alpha \cos(\lambda L) &= -\beta\left(\cos(\lambda L) + \frac{\alpha}{\lambda}\sin(\lambda L)\right) \\
\left(\lambda - \frac{\alpha\beta}{\lambda}\right)\sin(\lambda L) &= (\alpha+\beta)\cos(\lambda L)
\end{aligned}
$$

Note:

$$\cos(\lambda L) = 0 \iff \lambda = \frac{(2n-1)\pi}{2L}$$

with $n \in \mathbb{N}$.

Solution (8 of 10)

Suppose $\lambda = \dfrac{(2n-1)\pi}{2L}$ with $n \in \mathbb{N}$, then $\sin(\lambda L) = \pm 1$ and

$$\left(\lambda - \frac{\alpha\beta}{\lambda}\right)\sin(\lambda L) = (\alpha+\beta)\cos(\lambda L)$$
$$\lambda - \frac{\alpha\beta}{\lambda} = 0$$
$$\alpha\beta = \left(\frac{(2n-1)\pi}{2L}\right)^2.$$

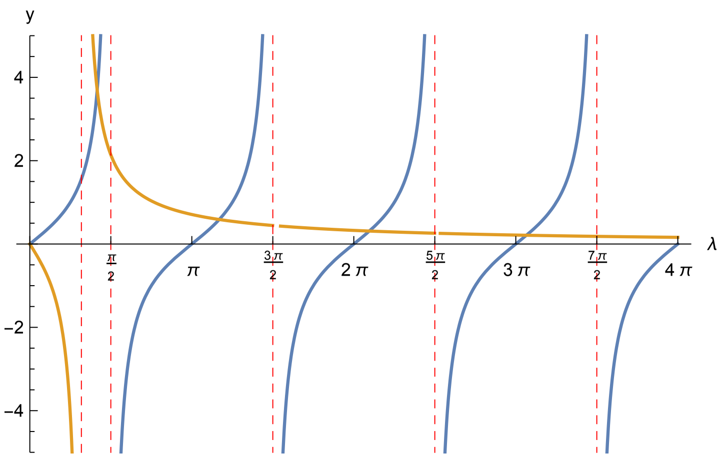
If there is no $n \in \mathbb{N}$ satisfying the last equation, then $\cos(\lambda L) \neq 0$ and

$$\left(\lambda - \frac{\alpha\beta}{\lambda}\right)\sin(\lambda L) = (\alpha+\beta)\cos(\lambda L)$$
$$\left(\lambda - \frac{\alpha\beta}{\lambda}\right)\tan(\lambda L) = \alpha+\beta$$
$$\tan(\lambda L) = \frac{\alpha+\beta}{\lambda - \frac{\alpha\beta}{\lambda}} = \frac{(\alpha+\beta)\lambda}{\lambda^2 - \alpha\beta}.$$

Solution (9 of 10)

$$\tan(\lambda L) = \frac{(\alpha+\beta)\lambda}{\lambda^2 - \alpha\beta}$$

This equation has positive solutions which can be approximated using Newton's method.



$L = \alpha = \beta = 1$

Solution (10 of 10)

The first few solutions are

$$\lambda_1 \approx 1.20779$$
$$\lambda_2 \approx 3.44824$$
$$\lambda_3 \approx 6.44095$$
$$\lambda_4 \approx 9.53048$$

and $\lambda_n \to (n-1)\pi$ as $n \to \infty$.
The corresponding eigenfunctions are

$$X_n(x) = A_n \cos(\lambda_n x) + B_n \sin(\lambda_n x)$$

while the product solutions have the form

$$u_n(x,t) = e^{-\lambda_n^2 t}\left(A_n \cos(\lambda_n x) + B_n \sin(\lambda_n x)\right).$$

Nonhomogeneous Boundary Conditions

Consider the IBVP:

$$u_t = k u_{xx} \quad \text{for } 0 < x < L,\ t > 0$$
$$u(0,t) = u_0$$
$$u(L,t) = u_L$$
$$u(x,0) = f(x)$$

Recall that the temperature in the region asymptotically approached a steady state of 0 when the boundary conditions were homogeneous ($u(0,t) = u(L,t) = 0$). Suppose that in the nonhomogeneous case the temperature distribution approaches a nonzero steady state $U(x)$.

Steady-State Solution

Suppose

$$\begin{aligned} U''(x) &= 0 \quad \text{for } 0 < x < L \\ U(0) &= u_0 \\ U(L) &= u_L \end{aligned}$$

then $U(x) = Ax + B$.

$$U(0) = u_0 = A(0) + B \implies B = u_0$$

The other boundary condition implies

$$U(L) = u_L = AL + u_0 \implies A = \frac{u_L - u_0}{L}.$$

Hence the steady-state solution is

$$U(x) = \frac{(u_L - u_0)x}{L} + u_0.$$

Time-Dependent Solution (1 of 2)

Assume the solution to the IBVP can be written as $u(x,t) = U(x) + v(x,t)$ where $v(x,t)$ is an unknown function.

Question: what IBVP must $v(x,t)$ satisfy?

$$\begin{aligned} u_t &= ku_{xx} \\ (U(x) + v(x,t))_t &= k(U(x) + v(x,t))_{xx} \\ v_t &= kv_{xx} \quad \text{(PDE)} \end{aligned}$$

$$\begin{aligned} u(0,t) &= u_0 \\ U(0) + v(0,t) &= u_0 \\ v(0,t) &= 0 \quad \text{(BC at } x = 0) \end{aligned}$$

Likewise $v(L,t) = 0$.

$$\begin{aligned} u(x,0) &= f(x) \\ U(x) + v(x,0) &= f(x) \\ v(x,0) &= f(x) - U(x) \quad \text{(IC at } t = 0) \end{aligned}$$

Time-Dependent Solution (2 of 2)

$$\begin{aligned} v_t &= kv_{xx} \quad \text{for } 0 < x < L \text{ and } t > 0 \\ v(0,t) &= 0 \\ v(L,t) &= 0 \\ v(x,0) &= f(x) - U(x) \end{aligned}$$

If $b_n = \dfrac{2}{L}\displaystyle\int_0^L (f(x) - U(x)) \sin \frac{n\pi x}{L}\, dx,$

then $v(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L},$

and $u(x,t) = u_0 + \dfrac{(u_L - u_0)x}{L} + \displaystyle\sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}.$

Example

Solve the following IBVP:

$$\begin{aligned} u_t &= 5u_{xx} \quad \text{for } 0 < x < 1 \text{ and } t > 0 \\ u(0,t) &= 10 \\ u(1,t) &= 20 \\ u(x,0) &= 10 + 11x - x^2 \end{aligned}$$

Solution (1 of 2)

The steady-state solution is

$$U(x) = 10 + \frac{(20-10)x}{1} = 10(1+x).$$

The transient solution $v(x,t)$ satisfies the IBVP:

$$\begin{aligned} v_t &= 5v_{xx} \quad \text{for } 0 < x < 1 \text{ and } t > 0 \\ v(0,t) &= 0 \\ v(1,t) &= 0 \\ v(x,0) &= 10 + 11x - x^2 - 10(1+x) = x - x^2 \end{aligned}$$

Solution (2 of 2)

$$\begin{aligned} b_n &= 2\int_0^1 (x - x^2)\sin(n\pi x)\,dx \\ &= \frac{4(1-(-1)^n)}{n^3\pi^3} \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ 8/(n^3\pi^3) & \text{if } n \text{ is odd.} \end{cases} \\ v(x,t) &= \frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{e^{-5(2n-1)^2\pi^2 t}}{(2n-1)^3}\sin((2n-1)\pi x) \\ u(x,t) &= 10(1+x) + \frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{e^{-5(2n-1)^2\pi^2 t}}{(2n-1)^3}\sin((2n-1)\pi x) \end{aligned}$$

Time Dependent Boundary Conditions

Consider the IBVP:

$$\begin{aligned} u_t &= ku_{xx}, \quad \text{for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= a(t) \\ u(L,t) &= b(t) \\ u(x,0) &= f(x) \end{aligned}$$

- In this situation the boundary conditions are functions of time.
- There may not be a steady-state solution, but the approach used in the case of constant, nonhomogeneous BCs is useful.

Define a **reference function** $r(x,t) = a(t) + \frac{(b(t)-a(t))x}{L}$ and suppose $u(x,t) = r(x,t) + v(x,t)$.

Question: what IBVP does $v(x,t)$ satisfy?

PDE for $v(x,t)$

$$\begin{aligned} u_t &= ku_{xx} \\ (r(x,t) + v(x,t))_t &= k(r(x,t) + v(x,t))_{xx} \\ a'(t) + \frac{(b'(t)-a'(t))x}{L} + v_t &= kv_{xx} \\ v_t &= kv_{xx} - a'(t) - \frac{(b'(t)-a'(t))x}{L} \end{aligned}$$

Note: this PDE is nonhomogeneous.

BCs for $v(x,t)$

$$\begin{aligned}
u(0,t) &= a(t)\\
r(0,t)+v(0,t) &= a(t)\\
a(t)+v(0,t) &= a(t)\\
v(0,t) &= 0 \quad \text{(BC at } x=0\text{)}\\
u(L,t) &= b(t)\\
r(L,t)+v(L,t) &= b(t)\\
b(t)+v(L,t) &= b(t)\\
v(L,t) &= 0 \quad \text{(BC at } x=L\text{)}
\end{aligned}$$

The boundary conditions are once again homogeneous.

IC for $v(x,t)$

$$\begin{aligned}
u(x,0) &= f(x)\\
r(x,0)+v(x,0) &= f(x)\\
a(0)+\frac{(b(0)-a(0))x}{L}+v(x,0) &= f(x)\\
v(x,0) &= f(x)-a(0)-\frac{(b(0)-a(0))x}{L}
\end{aligned}$$

IBVP for $v(x,t)$

In summary:

$$\begin{aligned}
v_t &= kv_{xx}-a'(t)-\frac{(b'(t)-a'(t))x}{L} \quad \text{for } 0<x<L \text{ and } t>0\\
v(0,t) &= 0\\
v(L,t) &= 0\\
v(x,0) &= f(x)-a(0)-\frac{(b(0)-a(0))x}{L}
\end{aligned}$$

Remarks:

- This IBVP has a nonhomogeneous PDE and a non-trivial initial condition.
- Search for a solution $v(x,t)=v_1(x,t)+v_2(x,t)$ for which $v_1(x,t)$ solves the homogeneous PDE with a non-zero initial condition and $v_2(x,t)$ solves a nonhomogeneous PDE with a zero initial condition.

Sum of Solutions (1 of 2)

Consider the two IBVPs with $0<x<L$ and $t>0$:

$$\begin{aligned}
(v_1)_t &= k(v_1)_{xx}\\
v_1(0,t) &= 0\\
v_1(L,t) &= 0\\
v_1(x,0) &= f(x)-\frac{(b(0)-a(0))x}{L}-a(0)
\end{aligned}$$

$$\begin{aligned}
(v_2)_t &= k(v_2)_{xx}-a'(t)-\frac{(b'(t)-a'(t))x}{L}\\
v_2(0,t) &= 0\\
v_2(L,t) &= 0\\
v_2(x,0) &= 0
\end{aligned}$$

Sum of Solutions (2 of 2)

If $v_1(x,t)$ solves the 1st IBVP and $v_2(x,t)$ solves the 2nd IBVP, then $v(x,t) = v_1(x,t) + v_2(x,t)$ solves:

$$\begin{aligned} v_t &= kv_{xx} - a'(t) - \frac{(b'(t) - a'(t))x}{L} \quad \text{for } 0 < x < L \text{ and } t > 0 \\ v(0,t) &= 0 \\ v(L,t) &= 0 \\ v(x,0) &= f(x) - a(0) - \frac{(b(0) - a(0))x}{L} \end{aligned}$$

Solution of IBVP with Homogeneous PDE

The IBVP,

$$\begin{aligned} (v_1)_t &= k(v_1)_{xx} \text{ for } 0 < x < L,\ t > 0 \\ v_1(0,t) &= 0 \\ v_1(L,t) &= 0 \\ v_1(x,0) &= f(x) - \frac{(b(0) - a(0))x}{L} - a(0) \end{aligned}$$

has solution

$$v_1(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L} \int_0^L \left(f(x) - \frac{(b(0) - a(0))x}{L} - a(0) \right) \sin \frac{n\pi x}{L}\, dx.$$

Solution of IBVP with Nonhomogeneous PDE

We will assume the solution to the IBVP:

$$\begin{aligned} (v_2)_t &= k(v_2)_{xx} - a'(t) - \frac{(b'(t) - a'(t))x}{L} \text{ for } 0 < x < L,\ t > 0 \\ v_2(0,t) &= 0 \\ v_2(L,t) &= 0 \\ v_2(x,0) &= 0 \end{aligned}$$

is of the form

$$v_2(x,t) = \sum_{n=1}^{\infty} b_n(t) e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L}$$

where the coefficients b_n are functions of t. Differentiate this solution and substitute it into the nonhomogeneous PDE.

Verifying the Solution

$$\begin{aligned} (v_2)_{xx} &= -\sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2} b_n(t) e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L} \\ (v_2)_t &= \sum_{n=1}^{\infty} \left(b_n'(t) e^{-kn^2\pi^2 t/L^2} - \frac{kn^2\pi^2}{L^2} b_n(t) e^{-kn^2\pi^2 t/L^2} \right) \sin \frac{n\pi x}{L} \end{aligned}$$

Since $(v_2)_t - k(v_2)_{xx} = -a'(t) - \frac{(b'(t) - a'(t))x}{L}$ then

$$\sum_{n=1}^{\infty} b_n'(t) e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi x}{L} = -a'(t) - \frac{(b'(t) - a'(t))x}{L}.$$

Multiply both sides by $\sin(m\pi x/L)$ and integrate over $[0, L]$.

Integration and Orthogonality

$$\begin{aligned}\frac{L}{2}b_m'(t)e^{-km^2\pi^2 t/L^2} &= -\int_0^L \left(a'(t) + \frac{(b'(t)-a'(t))x}{L}\right)\sin\frac{m\pi x}{L}\,dx \\ &= -\frac{L}{m\pi}(a'(t) - (-1)^m b'(t)) \\ b_m'(t) &= \frac{2}{m\pi}e^{km^2\pi^2 t/L^2}((-1)^m b'(t) - a'(t))\end{aligned}$$

Integrate both sides with respect to t.

$$\begin{aligned}\int_0^t b_m'(s)\,ds &= \frac{2}{m\pi}\int_0^t e^{km^2\pi^2 s/L^2}((-1)^m b'(s) - a'(s))\,ds \\ b_m(t) - b_m(0) &= \frac{2}{m\pi}\int_0^t e^{km^2\pi^2 s/L^2}((-1)^m b'(s) - a'(s))\,ds\end{aligned}$$

Since $v_2(x,0) = 0$ then $b_m(0) = 0$ for $m \in \mathbb{N}$.

Assembling the Solution

$$\begin{aligned}r(x,t) &= a(t) + \frac{(b(t)-a(t))x}{L} \\ v_1(x,t) &= \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2}\sin\frac{n\pi x}{L} \\ v_2(x,t) &= \sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\int_0^t e^{-kn^2\pi^2(t-s)/L^2}((-1)^n b'(s) - a'(s))\,ds\right]\sin\frac{n\pi x}{L} \\ u(x,t) &= r(x,t) + v_1(x,t) + v_2(x,t)\end{aligned}$$

Example

Find the solution to the following IBVP.

$$\begin{aligned}u_t &= u_{xx} \text{ for } 0 < x < 1 \text{ and } t > 0 \\ u(0,t) &= t \\ u(1,t) &= 1 + \sin t \\ u(x,0) &= x\end{aligned}$$

The reference function is

$$r(x,t) = t + (1 - t + \sin t)x.$$

Associated Homogeneous IVBP

$$\begin{aligned}(v_1)_t &= (v_1)_{xx} \quad \text{for } 0 < x < 1 \text{ and } t > 0 \\ v_1(0,t) &= 0 \\ v_1(1,t) &= 0 \\ v_1(x,0) &= x - (0 + (1 - 0 + \sin 0)x) = x - x = 0\end{aligned}$$

Hence $v_1(x,t) = 0$.

Associated Nonhomogeneous IVBP

$$
\begin{aligned}
(v_2)_t &= (v_2)_{xx} - 1 + (1 - \cos t)x \quad \text{for } 0 < x < 1 \text{ and } t > 0 \\
v_2(0,t) &= 0 \\
v_2(1,t) &= 0 \\
v_2(x,0) &= 0
\end{aligned}
$$

Assume solution $v_2(x,t)$ has the form

$$v_2(x,t) = \sum_{n=1}^{\infty} b_n(t) e^{-n^2\pi^2 t} \sin(n\pi x)$$

where

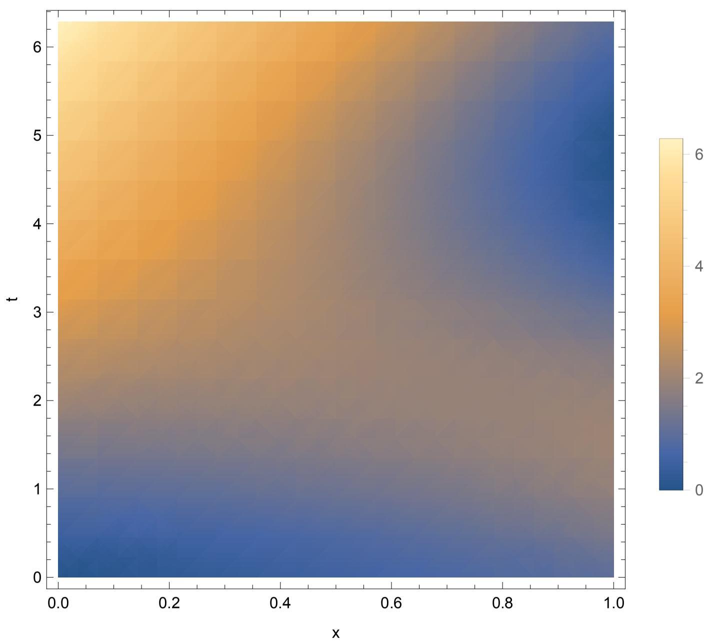
$$b_n(t) = \frac{2}{n\pi} \int_0^t e^{n^2\pi^2 s}((-1)^n \cos s - 1)\, ds.$$

Calculation of $b_n(t)$

$$
\begin{aligned}
b_n(t) &= \frac{2}{n\pi} \int_0^t e^{n^2\pi^2 s}((-1)^n \cos s - 1)\, ds \\
&= \frac{1 - e^{n^2\pi^2 t}}{n^2\pi^2} - \frac{(-1)^n n^2\pi^2}{1 + n^4\pi^4} + \frac{(-1)^n e^{n^2\pi^2 t}}{1 + n^4\pi^4}(n^2\pi^2 \cos t + \sin t)
\end{aligned}
$$

$$u(x,t) = t + (1 - t + \sin t)x + \sum_{n=1}^{\infty} b_n(t) e^{-n^2\pi^2 t} \sin(n\pi x)$$

Graph



Homework

- Read Section 4.2
- Exercises: 4, 5, 6