

# Heat Equation on Unbounded Intervals

MATH 467 *Partial Differential Equations*

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## Objectives

In this lesson we will learn about:

- ▶ the fundamental solution to the heat equation,
- ▶ solutions to the heat equation for  $0 \leq x < \infty$ , and
- ▶ solutions to the heat equation for  $-\infty < x < \infty$ .

# Fundamental Solution

For  $t > 0$  define the function

$$U(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}.$$

## Remarks:

- ▶  $U(x, t)$  is related to the probability density function for a normally distributed random variable.
- ▶ While defined only for  $t > 0$ , the limit as  $t \rightarrow 0^+$  exists.
- ▶  $U(x, t)$  solves the heat equation.

## Connection to Normal Distribution

A normally distributed, continuous random variable  $X$  with mean  $\mu$  and standard deviation  $\sigma$  has a probability distribution of

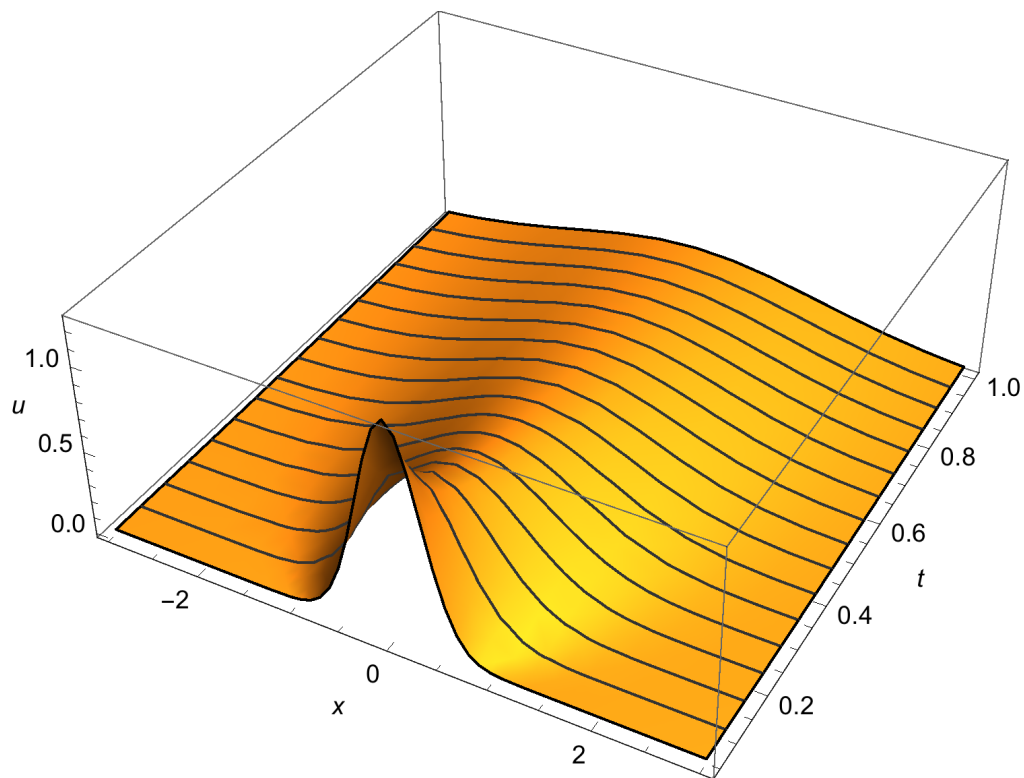
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \text{ for } -\infty < x < \infty.$$

Consider the fundamental solution to the heat equation,

$$U(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)} = \frac{1}{\sqrt{2kt} \sqrt{2\pi}} e^{-x^2/(2(\sqrt{2kt})^2)}.$$

For every  $t > 0$  the heat energy is distributed normally with mean  $\mu = 0$  and standard deviation  $\sigma = \sqrt{2kt}$ .

# Graph



$$\lim_{t \rightarrow 0^+} U(x, t)$$

If  $x \neq 0$  then

$$\lim_{t \rightarrow 0^+} U(x, t) = \lim_{t \rightarrow 0^+} \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)} = 0.$$

If  $x = 0$  then

$$\lim_{t \rightarrow 0^+} U(0, t) = \lim_{t \rightarrow 0^+} \frac{1}{\sqrt{4\pi kt}} = \infty.$$

## Justification

Suppose  $x \neq 0$  then

$$\begin{aligned}\lim_{t \rightarrow 0^+} U(x, t) &= \lim_{t \rightarrow 0^+} \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)} \\&= \lim_{t \rightarrow 0^+} \frac{1/\sqrt{t}}{\sqrt{4\pi k} e^{x^2/(4kt)}} \text{ (indeterminate } \infty/\infty) \\&= \lim_{t \rightarrow 0^+} \frac{-1/(2t^{3/2})}{-\frac{x^2}{4kt^2} \sqrt{4\pi k} e^{x^2/(4kt)}} \\&= \lim_{t \rightarrow 0^+} \frac{1}{\frac{x^2}{4kt^{1/2}} \sqrt{\pi k} e^{x^2/(4kt)}} \\&= 0.\end{aligned}$$

## Area Under the Curve

Assume that for fixed  $t > 0$  the improper integral

$$\int_{-\infty}^{\infty} U(x, t) dx$$

converges. Find the value of the integral.

## Solution

$$\begin{aligned}\text{If } S &= \int_{-\infty}^{\infty} U(x, t) dx \\ S^2 &= \left( \int_{-\infty}^{\infty} U(x, t) dx \right) \left( \int_{-\infty}^{\infty} U(y, t) dy \right) \\ &= \frac{1}{4\pi kt} \int_{-\infty}^{\infty} e^{-x^2/(4kt)} dx \int_{-\infty}^{\infty} e^{-y^2/(4kt)} dy \\ &= \frac{1}{4\pi kt} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/(4kt)} dx dy \\ &= \frac{1}{4\pi kt} \int_0^{2\pi} \int_0^{\infty} r e^{-r^2/(4kt)} dr d\theta \\ &= \frac{1}{2kt} \int_0^{\infty} r e^{-r^2/(4kt)} dr \\ S^2 &= 1 \implies S = 1.\end{aligned}$$

## Dirac Delta Function

Since

- ▶ for  $x \neq 0$ ,  $\lim_{t \rightarrow 0^+} U(x, t) = 0$  and
- ▶  $\int_{-\infty}^{\infty} U(x, t) dx = 1$  for all  $t > 0$

then

$$\lim_{t \rightarrow 0^+} \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)} = \delta(x)$$

the **Dirac delta function**.

## Logarithmic Differentiation (1 of 2)

$$U(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}$$

$$\ln U = -\frac{1}{2} \ln(4\pi kt) - \frac{x^2}{4kt}$$

$$\frac{\partial}{\partial t} [\ln U] = \frac{\partial}{\partial t} \left[ -\frac{1}{2} \ln(4\pi kt) - \frac{x^2}{4kt} \right]$$

$$\frac{U_t}{U} = -\frac{1}{2t} + \frac{x^2}{4kt^2}$$

$$U_t = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)} \left( -\frac{1}{2t} + \frac{x^2}{4kt^2} \right)$$

## Logarithmic Differentiation (2 of 2)

$$U(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}$$

$$\frac{\partial}{\partial x} [\ln U] = \frac{\partial}{\partial x} \left[ -\frac{1}{2} \ln(4\pi kt) - \frac{x^2}{4kt} \right]$$

$$\frac{U_x}{U} = -\frac{x}{2kt}$$

$$U_x = -U \left( \frac{x}{2kt} \right)$$

$$U_{xx} = -U_x \left( \frac{x}{2kt} \right) - U \left( \frac{1}{2kt} \right)$$

$$= U \left( \frac{x^2}{4k^2 t^2} \right) - U \left( \frac{1}{2kt} \right)$$

$$U_{xx} = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)} \left( \frac{x^2}{4k^2 t^2} - \frac{1}{2kt} \right)$$

# Solution to the Heat Equation

$$U_t(x, t) = \frac{1}{4\sqrt{k\pi t}} e^{-x^2/(4kt)} \left( \frac{x^2}{2kt^2} - \frac{1}{t} \right)$$
$$U_{xx}(x, t) = \frac{1}{4k\sqrt{k\pi t}} e^{-x^2/(4kt)} \left( \frac{x^2}{2kt^2} - \frac{1}{t} \right)$$

and thus  $U_t = kU_{xx}$ .

**Remark:** since the fundamental solution is defined for  $-\infty < x < \infty$ , no boundary conditions need be considered.

If  $u(x, 0) = f(x)$  is an initial condition defined on  $-\infty < x < \infty$ , how do we form a solution to the IVP?

## Solving the IVP

### Theorem

*Consider the initial value problem*

$$u_t = k u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0$$
$$u(x, 0) = f(x), \text{ for } -\infty < x < \infty.$$

*If  $f(x)$  is continuous and if  $\int_{-\infty}^{\infty} |f(x)| dx$  converges, then the piecewise defined function*

$$u(x, t) = \begin{cases} \int_{-\infty}^{\infty} U(x-y, t) f(y) dy & \text{if } t > 0, \\ f(x) & \text{if } t = 0 \end{cases}$$

*solves the heat equation and satisfies the initial condition in the sense that*

$$\lim_{(x,t) \rightarrow (x_0, 0^+)} u(x, t) = f(x_0).$$

# Uniqueness

## Theorem

Consider the initial value problem with conditions imposed as  $x \rightarrow \pm\infty$ ,

$$u_t = ku_{xx}, \text{ for } -\infty < x < \infty \text{ and } t > 0$$
$$u(x, 0) = f(x), \text{ for } -\infty < x < \infty$$

$$\lim_{x \rightarrow \pm\infty} \left( \max_{0 \leq t \leq T} |u(x, t)| \right) = 0$$

for  $-\infty < x < \infty$ ,  $t > 0$ , and  $T > 0$ . If  $f(x)$  is continuous, if

$\lim_{x \rightarrow \pm\infty} f(x) = 0$ , and if  $\int_{-\infty}^{\infty} |f(x)| dx$  converges, then

$$u(x, t) = \begin{cases} \int_{-\infty}^{\infty} U(x - y, t) f(y) dy & \text{if } t > 0, \\ f(x) & \text{if } t = 0 \end{cases}$$

is the unique, continuous solution to the initial value problem above on  $(-\infty, \infty) \times [0, \infty)$ .

## Example

Find the unique solution to the following initial boundary value problem.

$$u_t = u_{xx}, \text{ for } -\infty < x < \infty \text{ and } t > 0$$
$$u(x, 0) = e^{-x^2} \cos x, \text{ for } -\infty < x < \infty$$

$$\lim_{x \rightarrow \pm\infty} \left( \max_{0 \leq t \leq T} |u(x, t)| \right) = 0$$



## Solution (1 of 4)

Since  $k = 1$  then

$$\begin{aligned}u(x, t) &= \int_{-\infty}^{\infty} U(x - y, t) e^{-y^2} \cos y \, dy \\&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/(4t)} e^{-y^2} \operatorname{Re}(e^{iy}) \, dy \\&= \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)} \int_{-\infty}^{\infty} e^{(2xy-y^2)/(4t)} e^{-y^2} \operatorname{Re}(e^{iy}) \, dy \\&= U(x, t) \int_{-\infty}^{\infty} e^{(2xy-[1+4t]y^2)/(4t)} \operatorname{Re}(e^{iy}) \, dy \\&= U(x, t) \int_{-\infty}^{\infty} \operatorname{Re}\left(e^{(2xy-[1+4t]y^2)/(4t)} e^{iy}\right) \, dy \\u(x, t) &= U(x, t) \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{([2x+i4t]y-[1+4t]y^2)/(4t)} \, dy\right)\end{aligned}$$

## Solution (2 of 4)

$$u(x, t) = U(x, t) \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{([2x+i4t]y-[1+4t]y^2)/(4t)} \, dy\right)$$

Complete the square in the exponent.

$$\begin{aligned}u(x, t) &= U(x, t) \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{-\frac{1+4t}{4t}\left(y^2 - \frac{2x+i4t}{1+4t}y\right)} \, dy\right) \\&= U(x, t) \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{-\frac{1+4t}{4t}\left(y^2 - \frac{2x+i4t}{1+4t}y + \left[\frac{x+i2t}{1+4t}\right]^2 - \left[\frac{x+i2t}{1+4t}\right]^2\right)} \, dy\right) \\&= U(x, t) \operatorname{Re}\left(e^{\frac{1+4t}{4t}\left[\frac{x+i2t}{1+4t}\right]^2} \int_{-\infty}^{\infty} e^{-\frac{1+4t}{4t}\left(y - \frac{x+i2t}{1+4t}\right)^2} \, dy\right)\end{aligned}$$

## Solution (3 of 4)

$$u(x, t) = U(x, t) \operatorname{Re} \left( e^{\frac{(x+i2t)^2}{4t(1+4t)}} \int_{-\infty}^{\infty} e^{-\frac{1+4t}{2t} \left(y - \frac{x+i2t}{1+4t}\right)^2 / 2} dy \right)$$

Substitute  $z = \sqrt{\frac{1+4t}{2t}} \left(y - \frac{x+i2t}{1+4t}\right)$ .

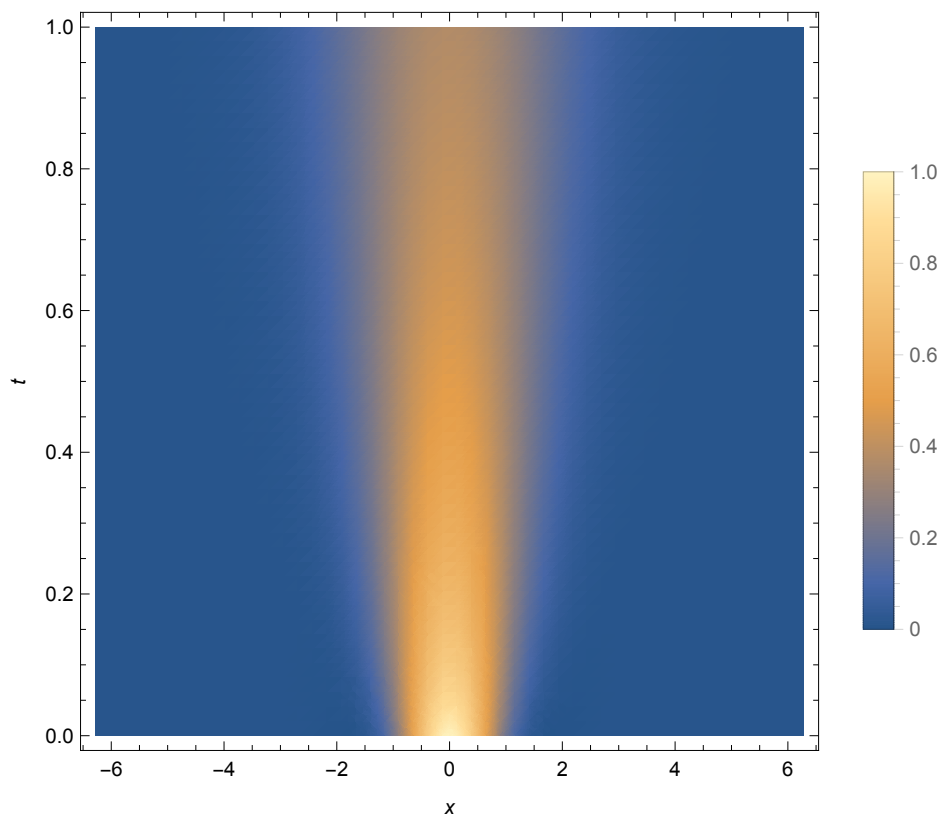
$$\begin{aligned} u(x, t) &= U(x, t) \operatorname{Re} \left( e^{\frac{(x+i2t)^2}{4t(1+4t)}} \sqrt{\frac{2t}{1+4t}} \int_{-\infty}^{\infty} e^{-z^2/2} dz \right) \\ &= U(x, t) \operatorname{Re} \left( e^{\frac{(x+i2t)^2}{4t(1+4t)}} \sqrt{\frac{4\pi t}{1+4t}} \right) \\ &= \sqrt{\frac{4\pi t}{1+4t}} U(x, t) \operatorname{Re} \left( e^{\frac{x^2+i4xt-4t^2}{4t(1+4t)}} \right) \end{aligned}$$

## Solution (4 of 4)

$$\begin{aligned} u(x, t) &= \sqrt{\frac{4\pi t}{1+4t}} U(x, t) \operatorname{Re} \left( e^{\frac{x^2-4t^2}{4t(1+4t)}} e^{\frac{i4xt}{4t(1+4t)}} \right) \\ &= \sqrt{\frac{4\pi t}{1+4t}} U(x, t) e^{\frac{x^2-4t^2}{4t(1+4t)}} \cos \left( \frac{x}{1+4t} \right) \\ &= \sqrt{\frac{4\pi t}{1+4t}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} e^{\frac{x^2-4t^2}{4t(1+4t)}} \cos \left( \frac{x}{1+4t} \right) \\ &= \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2+t}{1+4t}} \cos \left( \frac{x}{1+4t} \right) \end{aligned}$$

## Illustration

$$u(x, t) = \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2+t}{1+4t}} \cos\left(\frac{x}{1+4t}\right)$$



## Semi-Infinite Intervals

### Theorem

Suppose  $f(x)$  is continuous on  $[0, \infty)$ ,  $f(0) = 0$ ,  $\lim_{x \rightarrow \infty} f(x) = 0$ , and

$\int_0^\infty |f(x)| dx$  converges. The initial boundary value problem

$$u_t = k u_{xx} \text{ for } 0 < x < \infty \text{ and } t > 0$$

$$u(0+, t) = 0$$

$$u(x, 0+) = f(x)$$

$$\lim_{x \rightarrow \infty} \left( \max_{0 \leq t \leq T} |u(x, t)| \right) = 0$$

has a unique, continuous solution defined for  $t > 0$ ,

$$u(x, t) = \int_0^\infty (U(x-y, t) - U(x+y, t)) f(y) dy.$$

## Example

Solve the initial, boundary value problem:

$$u_t = u_{xx} \text{ for } 0 < x < \infty \text{ and } t > 0$$

$$u(0+, t) = 0$$

$$u(x, 0+) = x e^{-x}$$

$$\lim_{x \rightarrow \infty} \left( \max_{0 \leq t \leq T} |u(x, t)| \right) = 0.$$

## Solution (1 of 3)

For simplicity  $k = 1$  and thus

$$\begin{aligned} u(x, t) &= \int_0^\infty \frac{1}{\sqrt{4\pi t}} \left[ e^{-(x-y)^2/(4t)} - e^{-(x+y)^2/(4t)} \right] y e^{-y} dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty y \left[ e^{\frac{-(x^2 - 2(x-2t)y + y^2)}{4t}} - e^{\frac{-(x^2 + 2(x+2t)y + y^2)}{4t}} \right] dy \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \int_0^\infty y \left[ e^{\frac{-(y^2 - 2(x-2t)y)}{4t}} - e^{\frac{-(y^2 + 2(x+2t)y)}{4t}} \right] dy \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} e^{\frac{(x-2t)^2}{4t}} \int_0^\infty y e^{\frac{-(y-(x-2t))^2}{4t}} dy \\ &\quad - \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} e^{\frac{(x+2t)^2}{4t}} \int_0^\infty y e^{\frac{-(y+(x+2t))^2}{4t}} dy \end{aligned}$$

## Solution (2 of 3)

$$\begin{aligned}
 u(x, t) &= \frac{1}{\sqrt{4\pi t}} e^{-x+t} \int_0^\infty (y - (x - 2t) + (x - 2t)) e^{\frac{-(y - (x - 2t))^2}{4t}} dy \\
 &\quad - \frac{1}{\sqrt{4\pi t}} e^{x+t} \int_0^\infty (y + (x + 2t) - (x + 2t)) e^{\frac{-(y + (x + 2t))^2}{4t}} dy \\
 &= \frac{1}{\sqrt{\pi}} e^{-x+t} \int_0^\infty \frac{y - (x - 2t)}{2\sqrt{t}} e^{\frac{-(y - (x - 2t))^2}{4t}} dy \\
 &\quad + \frac{(x - 2t)e^{-x+t}}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{1}{2} \frac{(y - (x - 2t))^2}{2t}} dy \\
 &\quad - \frac{1}{\sqrt{\pi}} e^{x+t} \int_0^\infty \frac{y + (x + 2t)}{2\sqrt{t}} e^{-\frac{(y + (x + 2t))^2}{4t}} dy \\
 &\quad + \frac{(x + 2t)e^{x+t}}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{1}{2} \frac{(y + (x + 2t))^2}{2t}} dy
 \end{aligned}$$

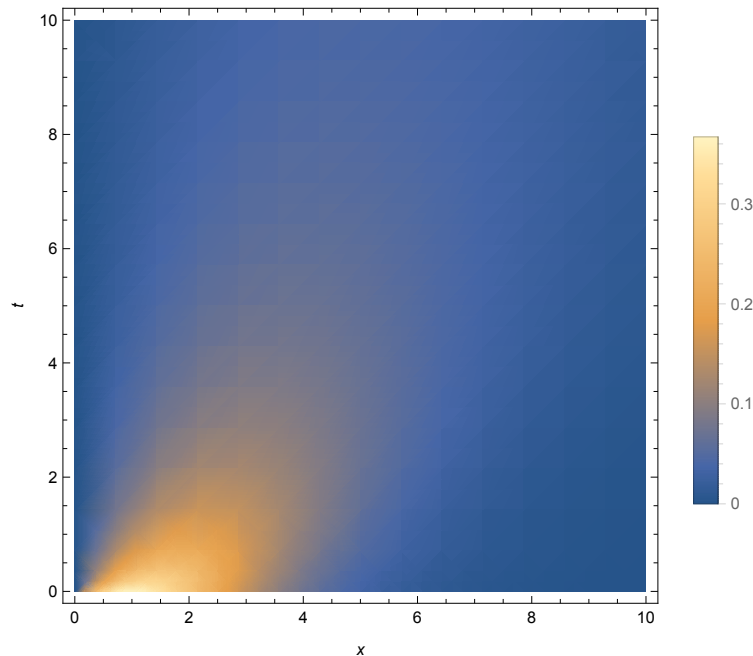
## Solution (3 of 3)

$$\begin{aligned}
 u(x, t) &= \sqrt{\frac{t}{\pi}} e^{-\frac{x^2}{4t}} + \frac{(x - 2t)e^{-x+t}}{\sqrt{2\pi}} \int_{-(x-2t)/\sqrt{2t}}^\infty e^{-\frac{z^2}{2}} dz \\
 &\quad - \sqrt{\frac{t}{\pi}} e^{-\frac{x^2}{4t}} + \frac{(x + 2t)e^{x+t}}{\sqrt{2\pi}} \int_{(x+2t)/\sqrt{2t}}^\infty e^{-\frac{z^2}{2}} dz \\
 &= (x - 2t)e^{-x+t} \left( 1 - \Phi \left( \frac{-(x - 2t)}{\sqrt{2t}} \right) \right) \\
 &\quad + (x + 2t)e^{x+t} \left( 1 - \Phi \left( \frac{x + 2t}{\sqrt{2t}} \right) \right)
 \end{aligned}$$

where  $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-y^2/2} dy$ .

## Illustration

$$u(x, t) = (x - 2t)e^{-x+t} \left(1 - \Phi \left(-(x - 2t)/\sqrt{2t}\right)\right) \\ + (x + 2t)e^{x+t} \left(1 - \Phi \left((x + 2t)/\sqrt{2t}\right)\right)$$



## Semi-Infinite Interval, Neumann BCs

### Theorem

Suppose  $f(x)$  is continuous on  $[0, \infty)$ ,  $f(0) = 0$ ,  $\lim_{x \rightarrow \infty} f(x) = 0$ , and

$\int_0^\infty |f(x)| dx$  converges. The initial boundary value problem

$$u_t = k u_{xx} \text{ for } 0 < x < \infty \text{ and } t > 0$$

$$u_x(0+, t) = 0$$

$$u(x, 0+) = f(x)$$

$$\lim_{x \rightarrow \infty} \left( \max_{0 \leq t \leq T} |u(x, t)| \right) = 0$$

has a unique, continuous solution defined for  $t > 0$ ,

$$u(x, t) = \int_0^\infty (U(x - y, t) + U(x + y, t)) f(y) dy.$$

## Example

Solve the initial, boundary value problem:

$$u_t = u_{xx} \text{ for } 0 < x < \infty \text{ and } t > 0$$

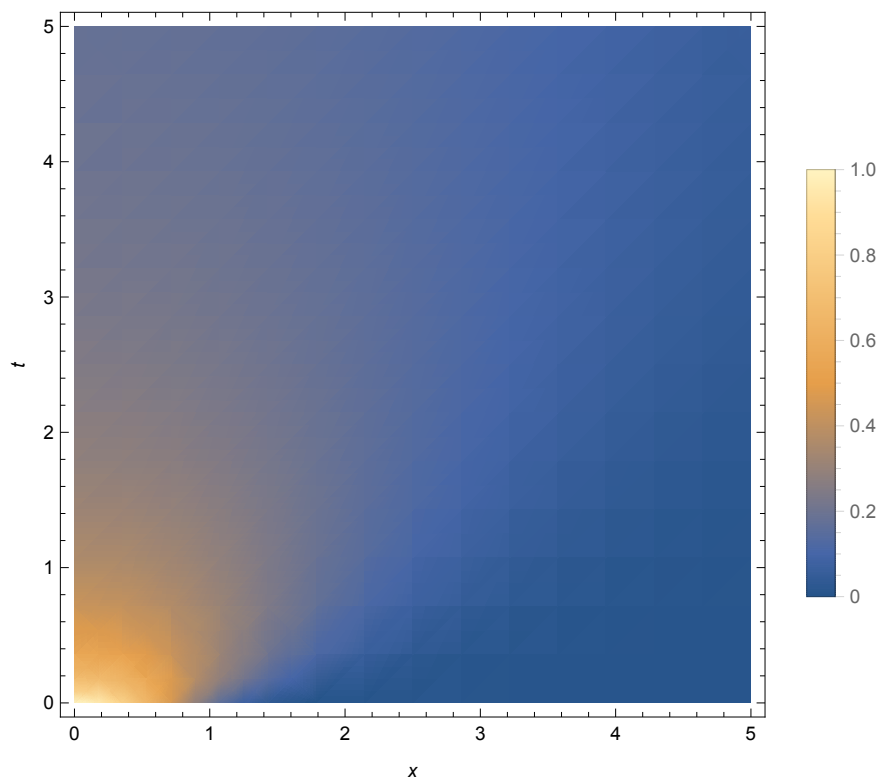
$$u_x(0+, t) = 0$$

$$u(x, 0+) = e^{-x^2} \cos x$$

$$\lim_{x \rightarrow \infty} \left( \max_{0 \leq t \leq T} |u(x, t)| \right) = 0.$$

## Solution

$$u(x, t) = \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2+t}{1+4t}} \cos \left( \frac{x}{1+4t} \right)$$



# Homework

- ▶ Read Section 4.4
- ▶ Exercises: 17, 18, 19