Heat Equation on Unbounded Intervals

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn about:

- the fundamental solution to the heat equation,
- solutions to the heat equation for $0 \leq x < \infty$, and
- solutions to the heat equation for $-\infty < x < \infty$.

Fundamental Solution

For $t > 0$ define the function

$$U(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/(4kt)}.$$

Remarks:

- $U(x,t)$ is related to the probability density function for a normally distributed random variable.
- While defined only for $t > 0$, the limit as $t \to 0^+$ exists.
- $U(x,t)$ solves the heat equation.

Connection to Normal Distribution

A normally distributed, continuous random variable X with mean μ and standard deviation σ has a probability distribution of

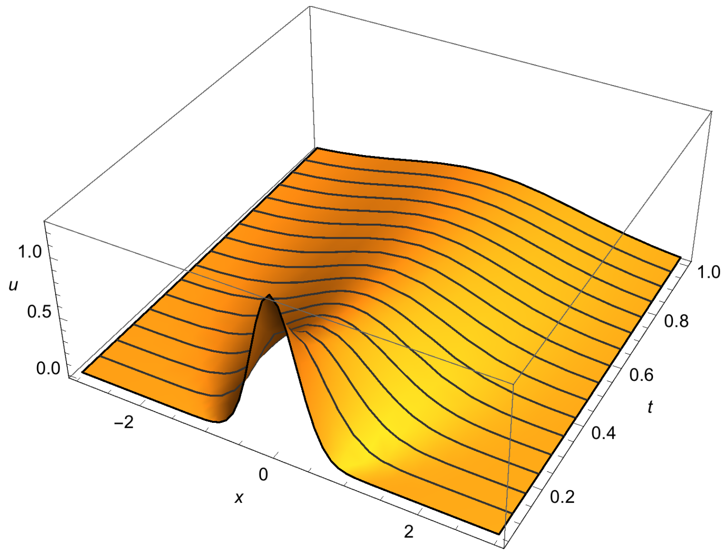
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \text{ for } -\infty < x < \infty.$$

Consider the fundamental solution to the heat equation,

$$U(x,t) = \frac{1}{\sqrt{4\pi k\,t}} e^{-x^2/(4kt)} = \frac{1}{\sqrt{2k\,t}\,\sqrt{2\pi}} e^{-x^2/(2(\sqrt{2k\,t})^2)}.$$

For every $t > 0$ the heat energy is distributed normally with mean $\mu = 0$ and standard deviation $\sigma = \sqrt{2k\,t}$.

Graph



$\lim_{t\to 0^+} U(x,t)$

If $x \neq 0$ then

$$\lim_{t\to 0^+} U(x,t) = \lim_{t\to 0^+} \frac{1}{\sqrt{4\pi k t}} e^{-x^2/(4kt)} = 0.$$

If $x = 0$ then

$$\lim_{t\to 0^+} U(0,t) = \lim_{t\to 0^+} \frac{1}{\sqrt{4\pi k t}} = \infty.$$

Justification

Suppose $x \neq 0$ then

$$\begin{aligned}
\lim_{t\to 0^+} U(x,t) &= \lim_{t\to 0^+} \frac{1}{\sqrt{4\pi k t}} e^{-x^2/(4kt)} \\
&= \lim_{t\to 0^+} \frac{1/\sqrt{t}}{\sqrt{4\pi k} e^{x^2/(4kt)}} \text{ (indeterminate } \infty/\infty) \\
&= \lim_{t\to 0^+} \frac{-1/(2t^{3/2})}{-\frac{x^2}{4kt^2}\sqrt{4\pi k} e^{x^2/(4kt)}} \\
&= \lim_{t\to 0^+} \frac{1}{\frac{x^2}{4kt^{1/2}}\sqrt{\pi k} e^{x^2/(4kt)}} \\
&= 0.
\end{aligned}$$

Area Under the Curve

Assume that for fixed $t > 0$ the improper integral

$$\int_{-\infty}^{\infty} U(x,t)\,dx$$

converges. Find the value of the integral.

Solution

$$\text{If } S = \int_{-\infty}^{\infty} U(x,t)\,dx$$

$$\begin{aligned}
S^2 &= \left(\int_{-\infty}^{\infty} U(x,t)\,dx\right)\left(\int_{-\infty}^{\infty} U(y,t)\,dy\right) \\
&= \frac{1}{4\pi kt}\int_{-\infty}^{\infty} e^{-x^2/(4kt)}\,dx \int_{-\infty}^{\infty} e^{-y^2/(4kt)}\,dy \\
&= \frac{1}{4\pi kt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/(4kt)}\,dx\,dy \\
&= \frac{1}{4\pi kt}\int_{0}^{2\pi}\int_{0}^{\infty} re^{-r^2/(4kt)}\,dr\,d\theta \\
&= \frac{1}{2kt}\int_{0}^{\infty} re^{-r^2/(4kt)}\,dr \\
S^2 &= 1 \implies S = 1.
\end{aligned}$$

Dirac Delta Function

Since

- for $x \neq 0$, $\lim_{t\to 0^+} U(x,t) = 0$ and
- $\int_{-\infty}^{\infty} U(x,t)\,dx = 1$ for all $t > 0$

then

$$\lim_{t\to 0^+} \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)} = \delta(x)$$

the **Dirac delta function.**

Logarithmic Differentiation (1 of 2)

$$\begin{aligned}
U(x,t) &= \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)} \\
\ln U &= -\frac{1}{2}\ln(4\pi kt) - \frac{x^2}{4kt} \\
\frac{\partial}{\partial t}[\ln U] &= \frac{\partial}{\partial t}\left[-\frac{1}{2}\ln(4\pi kt) - \frac{x^2}{4kt}\right] \\
\frac{U_t}{U} &= -\frac{1}{2t} + \frac{x^2}{4kt^2} \\
U_t &= \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}\left(-\frac{1}{2t} + \frac{x^2}{4kt^2}\right)
\end{aligned}$$

Logarithmic Differentiation (2 of 2)

$$\begin{aligned}
U(x,t) &= \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)} \\
\frac{\partial}{\partial x}[\ln U] &= \frac{\partial}{\partial x}\left[-\frac{1}{2}\ln(4\pi kt) - \frac{x^2}{4kt}\right] \\
\frac{U_x}{U} &= -\frac{x}{2kt} \\
U_x &= -U\left(\frac{x}{2kt}\right) \\
U_{xx} &= -U_x\left(\frac{x}{2kt}\right) - U\left(\frac{1}{2kt}\right) \\
&= U\left(\frac{x^2}{4k^2t^2}\right) - U\left(\frac{1}{2kt}\right) \\
U_{xx} &= \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}\left(\frac{x^2}{4k^2t^2} - \frac{1}{2kt}\right)
\end{aligned}$$

Solution to the Heat Equation

$$U_t(x,t) = \frac{1}{4\sqrt{k\pi t}} e^{-x^2/(4kt)} \left(\frac{x^2}{2kt^2} - \frac{1}{t} \right)$$
$$U_{xx}(x,t) = \frac{1}{4k\sqrt{k\pi t}} e^{-x^2/(4kt)} \left(\frac{x^2}{2kt^2} - \frac{1}{t} \right)$$

and thus $U_t = kU_{xx}$.

Remark: since the fundamental solution is defined for $-\infty < x < \infty$, no boundary conditions need be considered.

If $u(x,0) = f(x)$ is an initial condition defined on $-\infty < x < \infty$, how do we form a solution to the IVP?

Solving the IVP

Theorem

Consider the initial value problem

$$u_t = k\, u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0$$
$$u(x,0) = f(x), \text{ for } -\infty < x < \infty.$$

If $f(x)$ is continuous and if $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, then the piecewise defined function

$$u(x,t) = \begin{cases} \int_{-\infty}^{\infty} U(x-y,t) f(y)\, dy & \text{if } t > 0, \\ f(x) & \text{if } t = 0 \end{cases}$$

solves the heat equation and satisfies the initial condition in the sense that

$$\lim_{(x,t)\to(x_0,0^+)} u(x,t) = f(x_0).$$

Uniqueness

Theorem

Consider the initial value problem with conditions imposed as $x \to \pm\infty$,

$$u_t = ku_{xx}, \text{ for } -\infty < x < \infty \text{ and } t > 0$$
$$u(x,0) = f(x), \text{ for } -\infty < x < \infty$$
$$\lim_{x\to\pm\infty} \left(\max_{0\le t\le T} |u(x,t)| \right) = 0$$

for $-\infty < x < \infty$, $t > 0$, and $T > 0$. If $f(x)$ is continuous, if $\lim_{x\to\pm\infty} f(x) = 0$, and if $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, then

$$u(x,t) = \begin{cases} \int_{-\infty}^{\infty} U(x-y,t) f(y)\, dy & \text{if } t > 0, \\ f(x) & \text{if } t = 0 \end{cases}$$

is the unique, continuous solution to the initial value problem above on $(-\infty,\infty) \times [0,\infty)$.

Example

Find the unique solution to the following initial boundary value problem.

$$u_t = u_{xx}, \text{ for } -\infty < x < \infty \text{ and } t > 0$$
$$u(x,0) = e^{-x^2} \cos x, \text{ for } -\infty < x < \infty$$
$$\lim_{x\to\pm\infty} \left(\max_{0\le t\le T} |u(x,t)| \right) = 0$$

Solution (1 of 4)

Since $k = 1$ then

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} U(x-y,t) e^{-y^2} \cos y \, dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/(4t)} e^{-y^2} \operatorname{Re}\left(e^{iy}\right) \, dy \\
&= \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)} \int_{-\infty}^{\infty} e^{(2xy-y^2)/(4t)} e^{-y^2} \operatorname{Re}\left(e^{iy}\right) \, dy \\
&= U(x,t) \int_{-\infty}^{\infty} e^{(2xy-[1+4t]y^2)/(4t)} \operatorname{Re}\left(e^{iy}\right) \, dy \\
&= U(x,t) \int_{-\infty}^{\infty} \operatorname{Re}\left(e^{(2xy-[1+4t]y^2)/(4t)} e^{iy}\right) \, dy \\
u(x,t) &= U(x,t) \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{([2x+i4t]y-[1+4t]y^2)/(4t)} \, dy\right)
\end{aligned}$$

Solution (2 of 4)

$$u(x,t) = U(x,t) \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{([2x+i4t]y-[1+4t]y^2)/(4t)} \, dy\right)$$

Complete the square in the exponent.

$$\begin{aligned}
u(x,t) &= U(x,t) \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{-\frac{1+4t}{4t}\left(y^2 - \frac{2x+i4t}{1+4t} y\right)} \, dy\right) \\
&= U(x,t) \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{-\frac{1+4t}{4t}\left(y^2 - \frac{2x+i4t}{1+4t} y + \left[\frac{x+i2t}{1+4t}\right]^2 - \left[\frac{x+i2t}{1+4t}\right]^2\right)} \, dy\right) \\
&= U(x,t) \operatorname{Re}\left(e^{\frac{1+4t}{4t}\left[\frac{x+i2t}{1+4t}\right]^2} \int_{-\infty}^{\infty} e^{-\frac{1+4t}{4t}\left(y - \frac{x+i2t}{1+4t}\right)^2} \, dy\right)
\end{aligned}$$

Solution (3 of 4)

$$u(x,t) = U(x,t) \operatorname{Re}\left(e^{\frac{(x+i2t)^2}{4t(1+4t)}} \int_{-\infty}^{\infty} e^{-\frac{1+4t}{2t}\left(y - \frac{x+i2t}{1+4t}\right)^2/2} \, dy\right)$$

Substitute $z = \sqrt{\frac{1+4t}{2t}}\left(y - \frac{x+i2t}{1+4t}\right)$.

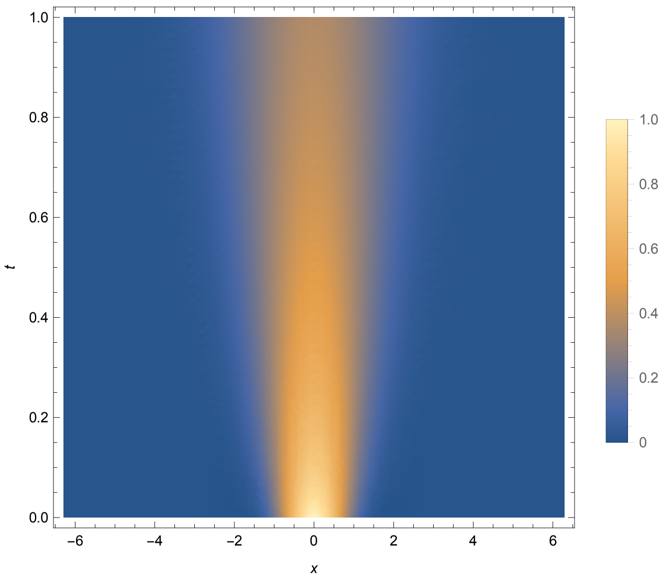
$$\begin{aligned}
u(x,t) &= U(x,t) \operatorname{Re}\left(e^{\frac{(x+i2t)^2}{4t(1+4t)}} \sqrt{\frac{2t}{1+4t}} \int_{-\infty}^{\infty} e^{-z^2/2} \, dz\right) \\
&= U(x,t) \operatorname{Re}\left(e^{\frac{(x+i2t)^2}{4t(1+4t)}} \sqrt{\frac{4\pi t}{1+4t}}\right) \\
&= \sqrt{\frac{4\pi t}{1+4t}} U(x,t) \operatorname{Re}\left(e^{\frac{x^2+i4xt-4t^2}{4t(1+4t)}}\right)
\end{aligned}$$

Solution (4 of 4)

$$\begin{aligned}
u(x,t) &= \sqrt{\frac{4\pi t}{1+4t}} U(x,t) \operatorname{Re}\left(e^{\frac{x^2-4t^2}{4t(1+4t)}} e^{\frac{i4xt}{4t(1+4t)}}\right) \\
&= \sqrt{\frac{4\pi t}{1+4t}} U(x,t) e^{\frac{x^2-4t^2}{4t(1+4t)}} \cos\left(\frac{x}{1+4t}\right) \\
&= \sqrt{\frac{4\pi t}{1+4t}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} e^{\frac{x^2-4t^2}{4t(1+4t)}} \cos\left(\frac{x}{1+4t}\right) \\
&= \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2+t}{1+4t}} \cos\left(\frac{x}{1+4t}\right)
\end{aligned}$$

Illustration

$$u(x,t) = \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2+t}{1+4t}} \cos\left(\frac{x}{1+4t}\right)$$



Semi-Infinite Intervals

Theorem

Suppose $f(x)$ is continuous on $[0,\infty)$, $f(0) = 0$, $\lim_{x\to\infty} f(x) = 0$, and $\int_0^\infty |f(x)|\,dx$ converges. The initial boundary value problem

$$\begin{aligned} u_t &= k\, u_{xx} \text{ for } 0 < x < \infty \text{ and } t > 0 \\ u(0+,t) &= 0 \\ u(x,0+) &= f(x) \\ \lim_{x\to\infty}\left(\max_{0\le t\le T} |u(x,t)|\right) &= 0 \end{aligned}$$

has a unique, continuous solution defined for $t > 0$,

$$u(x,t) = \int_0^\infty (U(x-y,t) - U(x+y,t))\, f(y)\, dy.$$

Example

Solve the initial, boundary value problem:

$$\begin{aligned} u_t &= u_{xx} \text{ for } 0 < x < \infty \text{ and } t > 0 \\ u(0+,t) &= 0 \\ u(x,0+) &= x\,e^{-x} \\ \lim_{x\to\infty}\left(\max_{0\le t\le T} |u(x,t)|\right) &= 0. \end{aligned}$$

Solution (1 of 3)

For simplicity $k = 1$ and thus

$$\begin{aligned} u(x,t) &= \int_0^\infty \frac{1}{\sqrt{4\pi t}} \left[e^{-(x-y)^2/(4t)} - e^{-(x+y)^2/(4t)}\right] y e^{-y}\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty y \left[e^{\frac{-(x^2-2(x-2t)y+y^2)}{4t}} - e^{\frac{-(x^2+2(x+2t)y+y^2)}{4t}}\right] dy \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \int_0^\infty y \left[e^{\frac{-(y^2-2(x-2t)y)}{4t}} - e^{\frac{-(y^2+2(x+2t)y)}{4t}}\right] dy \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} e^{\frac{(x-2t)^2}{4t}} \int_0^\infty y\, e^{\frac{-(y-(x-2t))^2}{4t}}\, dy \\ &\quad - \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} e^{\frac{(x+2t)^2}{4t}} \int_0^\infty y\, e^{\frac{-(y+(x+2t))^2}{4t}}\, dy \end{aligned}$$

Solution (2 of 3)

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi t}} e^{-x+t} \int_0^\infty (y-(x-2t)+(x-2t)) e^{\frac{-(y-(x-2t))^2}{4t}}\, dy \\
&\quad - \frac{1}{\sqrt{4\pi t}} e^{x+t} \int_0^\infty (y+(x+2t)-(x+2t)) e^{\frac{-(y+(x+2t))^2}{4t}}\, dy \\
&= \frac{1}{\sqrt{\pi}} e^{-x+t} \int_0^\infty \frac{y-(x-2t)}{2\sqrt{t}} e^{\frac{-(y-(x-2t))^2}{4t}}\, dy \\
&\quad + \frac{(x-2t)e^{-x+t}}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{1}{2}\frac{(y-(x-2t))^2}{2t}}\, dy \\
&\quad - \frac{1}{\sqrt{\pi}} e^{x+t} \int_0^\infty \frac{y+(x+2t)}{2\sqrt{t}} e^{-\frac{(y+(x+2t))^2}{4t}}\, dy \\
&\quad + \frac{(x+2t)e^{x+t}}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{1}{2}\frac{(y+(x+2t))^2}{2t}}\, dy
\end{aligned}
$$

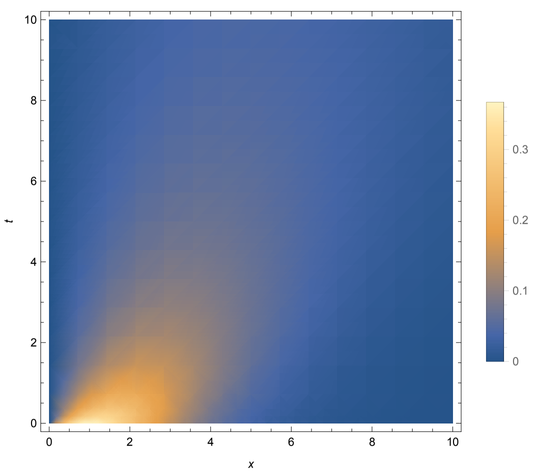
Solution (3 of 3)

$$
\begin{aligned}
u(x,t) &= \sqrt{\frac{t}{\pi}} e^{-\frac{x^2}{4t}} + \frac{(x-2t)e^{-x+t}}{\sqrt{2\pi}} \int_{-(x-2t)/\sqrt{2t}}^\infty e^{-\frac{z^2}{2}}\, dz \\
&\quad - \sqrt{\frac{t}{\pi}} e^{-\frac{x^2}{4t}} + \frac{(x+2t)e^{x+t}}{\sqrt{2\pi}} \int_{(x+2t)/\sqrt{2t}}^\infty e^{-\frac{z^2}{2}}\, dz \\
&= (x-2t)e^{-x+t}\left(1-\Phi\left(\frac{-(x-2t)}{\sqrt{2t}}\right)\right) \\
&\quad + (x+2t)e^{x+t}\left(1-\Phi\left(\frac{x+2t}{\sqrt{2t}}\right)\right)
\end{aligned}
$$

where $\Phi(y) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{y} e^{-y^2/2}\, dy.$

Illustration

$$
\begin{aligned}
u(x,t) &= (x-2t)e^{-x+t}\left(1-\Phi\left(-(x-2t)/\sqrt{2t}\right)\right) \\
&\quad + (x+2t)e^{x+t}\left(1-\Phi\left((x+2t)/\sqrt{2t}\right)\right)
\end{aligned}
$$



Semi-Infinite Interval, Neumann BCs

Theorem

Suppose $f(x)$ is continuous on $[0,\infty)$, $f(0)=0$, $\lim_{x\to\infty} f(x)=0$, and $\int_0^\infty |f(x)|\,dx$ converges. The initial boundary value problem

$$
\begin{aligned}
u_t &= k\, u_{xx} \text{ for } 0 < x < \infty \text{ and } t > 0 \\
u_x(0+,t) &= 0 \\
u(x,0+) &= f(x) \\
\lim_{x\to\infty}\left(\max_{0\le t\le T} |u(x,t)|\right) &= 0
\end{aligned}
$$

has a unique, continuous solution defined for $t>0$,

$$
u(x,t) = \int_0^\infty (U(x-y,t)+U(x+y,t))\, f(y)\, dy.
$$

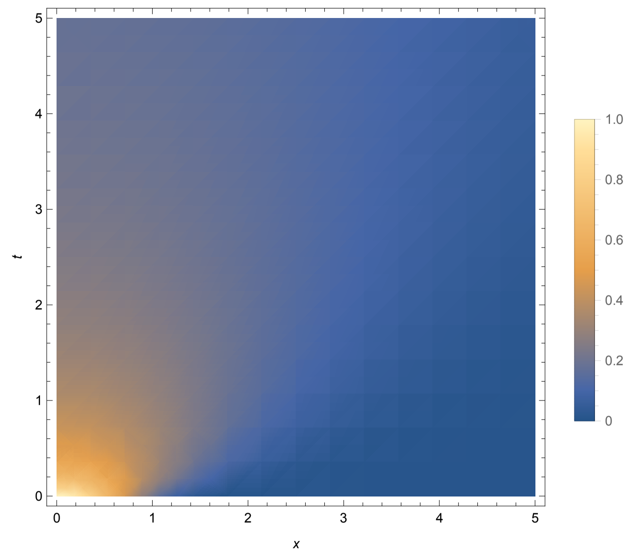
Example

Solve the initial, boundary value problem:

$$\begin{aligned} u_t &= u_{xx} \text{ for } 0 < x < \infty \text{ and } t > 0 \\ u_x(0+, t) &= 0 \\ u(x, 0+) &= e^{-x^2} \cos x \\ \lim_{x \to \infty} \left(\max_{0 \le t \le T} |u(x,t)| \right) &= 0. \end{aligned}$$

Solution

$$u(x,t) = \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2+t}{1+4t}} \cos\left(\frac{x}{1+4t}\right)$$



Homework

- Read Section 4.4
- Exercises: 17, 18, 19