$$u(x,t) = \int_{-\infty}^{\infty} U(x-y,t)\, e^{-y^2} \cos y \, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} e^{-y^2} \operatorname{Re}\left(e^{iy}\right) dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-x^2+2xy-y^2}{4t}} e^{-y^2} \operatorname{Re}\left(e^{iy}\right) dy$$

$$= \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \int_{-\infty}^{\infty} e^{\frac{-y^2+2xy}{4t}} e^{-\frac{4ty^2}{4t}} \operatorname{Re}\left(e^{iy}\right) dy$$

$$= \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \int_{-\infty}^{\infty} e^{\frac{-(1+4t)y^2+2xy}{4t}} \operatorname{Re}\left(e^{iy}\right) dy$$

$$= \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \int_{-\infty}^{\infty} \operatorname{Re}\left(e^{\frac{-(1+4t)y^2+2xy}{4t}} e^{i4ty/4t} \right) dy$$

$$= \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{\frac{-(1+4t)y^2+2(x+2it)y}{4t}} dy \right)$$

Complete the square for y in the exponential. First factor out $-(1+4t)$.

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{\frac{-(1+4t)\left(y^2 - \frac{2(x+2it)y}{1+4t}\right)}{4t}} dy \right)$$

Halve the coefficient of y, add and subtract its square.

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{\frac{-(1+4t)\left(y^2 - \frac{2(x+2it)}{1+4t}y + \frac{(x+2it)^2}{(1+4t)^2} - \frac{(x+2it)^2}{(1+4t)^2}\right)}{4t}} dy \right)$$

$$= \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \operatorname{Re}\left(\int_{-\infty}^{\infty} e^{\frac{-(1+4t)\left(y - \frac{x+2it}{1+4t}\right)^2}{4t}} e^{\frac{(x+2it)^2}{4t(1+4t)}} dy \right)$$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}\, \mathrm{Re}\left(\int_{-\infty}^{\infty} e^{\frac{-(1+4t)\left(y-\frac{x+2it}{1+4t}\right)^2}{4t}}\, e^{\frac{x^2+4xit-4t^2}{4t(1+4t)}}\, dy \right)$$

$$= \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}\, e^{\frac{x^2-4t^2}{4t(1+4t)}}\, \mathrm{Re}\left(\int_{-\infty}^{\infty} e^{\frac{-(1+4t)\left(y-\frac{x+2it}{1+4t}\right)^2}{4t}}\, e^{\frac{xi}{1+4t}}\, dy \right)$$

$$= \frac{1}{\sqrt{4\pi t}}\, e^{\frac{-x^2(1+4t)+x^2-4t^2}{4t(1+4t)}}\, \mathrm{Re}\left(e^{\frac{xi}{1+4t}} \int_{-\infty}^{\infty} e^{-\frac{(1+4t)}{4t}\left(y-\frac{x+2it}{1+4t}\right)^2} dy \right)$$

Substitute $\quad v = \frac{\sqrt{1+4t}}{\sqrt{2t}}\left(y - \frac{x+2it}{1+4t}\right)$

$$dv = \frac{\sqrt{1+4t}}{\sqrt{2t}}\, dy \quad \Rightarrow \quad dy = \frac{\sqrt{2t}}{\sqrt{1+4t}}\, dv$$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{\frac{-4x^2t-4t^2}{4t(1+4t)}}\, \mathrm{Re}\left(e^{\frac{xi}{1+4t}} \int_{-\infty}^{\infty} e^{-v^2/2}\, \frac{\sqrt{2t}}{\sqrt{1+4t}}\, dv \right)$$

$$= \frac{1}{\sqrt{4\pi t}}\, e^{\frac{-(x^2+t)}{1+4t}}\, \mathrm{Re}\left(e^{\frac{xi}{1+4t}}\, \frac{\sqrt{2t}}{\sqrt{1+4t}} \int_{-\infty}^{\infty} e^{-v^2/2}\, dv \right)$$

$$= \frac{\sqrt{2t}}{\sqrt{(1+4t)4\pi t}}\, e^{\frac{-(x^2+t)}{1+4t}}\, \mathrm{Re}\left(e^{\frac{xi}{1+4t}}\, \sqrt{2\pi} \right)$$

$$= \frac{1}{\sqrt{1+4t}}\, e^{\frac{-(x^2+t)}{1+4t}}\, \mathrm{Re}\left(e^{\frac{xi}{1+4t}} \right)$$

$$u(x,t) = \frac{1}{\sqrt{1+4t}}\, e^{\frac{-(x^2+t)}{1+4t}}\, \cos\left(\frac{x}{1+4t}\right)$$