

Maximum Principle and Uniqueness of Solutions

MATH 467 *Partial Differential Equations*

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The Maximum Principle

Theorem

Consider the initial boundary value problem

$$\begin{aligned} u_t &= k \, u_{xx}, & 0 < x < L, & \quad t > 0 \\ u(0, t) &= a(t) & \text{and} & \quad u(L, t) = b(t), & \quad t > 0 \\ u(x, 0) &= f(x), & 0 \leq x \leq L \end{aligned}$$

where the diffusion constant $k > 0$, and the functions $a(t)$, $b(t)$, and $f(x)$ are C^2 (twice continuously differentiable) on their respective intervals. Let $T > 0$ be a fixed time and let

$$A = \max_{0 \leq t \leq T} \{a(t)\}, \quad B = \max_{0 \leq t \leq T} \{b(t)\}, \quad \text{and} \quad F = \max_{0 \leq x \leq L} \{f(x)\}.$$

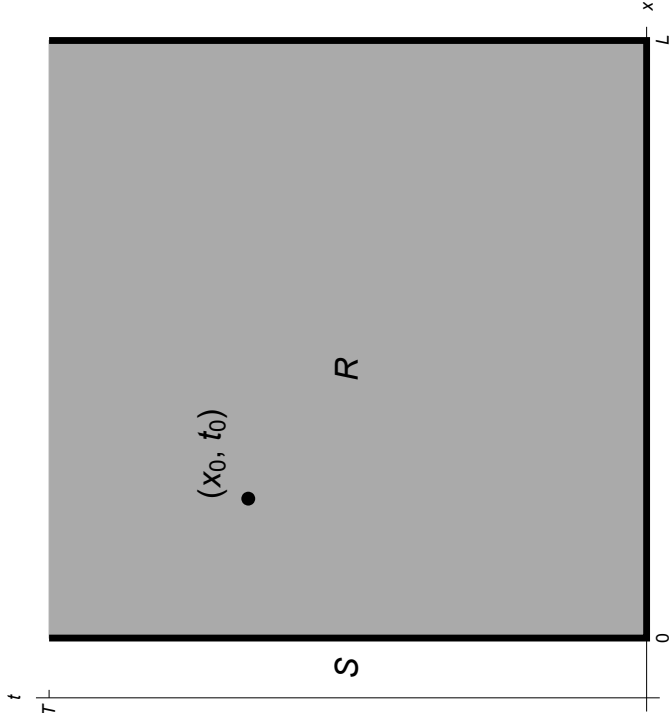
If $M = \max\{A, B, F\}$ and if $u(x, t)$ is any C^2 solution of the initial boundary value problem, then $u(x, t) \leq M$ for all $0 \leq x \leq L$ and $0 \leq t \leq T$.

Objectives

In this lesson we will explore:

- ▶ the Maximum Principle for solutions to the heat equation and its justification,
- ▶ the dependence of solutions to the heat equation on the initial and boundary conditions, and
- ▶ the uniqueness of solutions to the heat equation and its justification.

Illustration



Example

Consider the IBVP:

$$u_t = 9u_{xx} \text{ for } 0 < x < 3 \text{ and } t > 0$$

$$u(0, t) = 0 = u(3, t)$$

$$u(x, 0) = 6 \sin \frac{\pi x}{3} + 2 \sin \pi x$$

Find an upper bound for the solution.

Solution (1 of 3)

- ▶ According to the Maximum Principle, the maximum occurs either where $x = 0$, $x = 3$, or $t = 0$.
- ▶ Either the upper bound is $u(0, t) = u(3, t) = 0$ or the maximum occurs where

$$u(x, 0) = f(x) = 6 \sin \frac{\pi x}{3} + 2 \sin \pi x.$$

$$f'(x) = 2\pi \left(\cos \frac{\pi x}{3} + \cos \pi x \right)$$

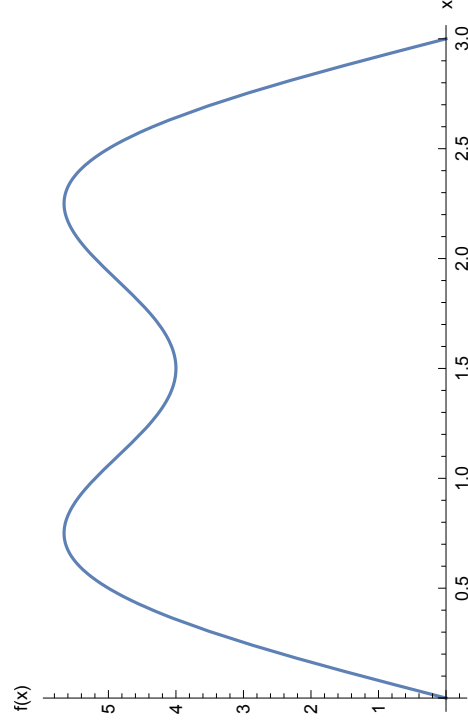
$$= 2\pi \left(\cos \left(\frac{2\pi x}{3} - \frac{\pi x}{3} \right) + \cos \left(\frac{2\pi x}{3} + \frac{\pi x}{3} \right) \right)$$

$$= 4\pi \cos \frac{\pi x}{3} \cos \frac{\pi x}{3}$$

Solution (2 of 3)

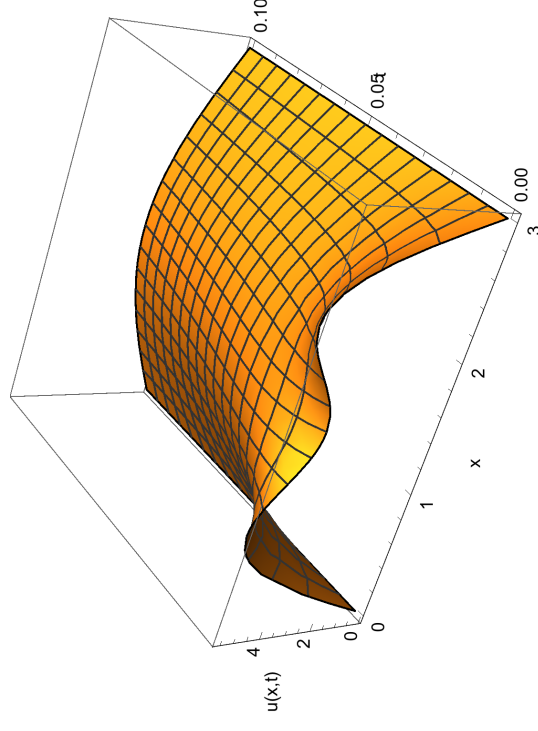
$$f'(x) = 4\pi \cos \frac{2\pi x}{3} \cos \frac{\pi x}{3} = 0$$

has critical numbers $x = 3/4$, $x = 9/4$ (both maxima) and $x = 3/2$ (minimum) in the interval $[0, 3]$.



Solution (3 of 3)

$$u(x, t) \leq f(3/4) = f(9/4) = 4\sqrt{2}$$



Minimum Principle

Corollary

Consider the initial boundary value problem

$$\begin{aligned} u_t &= ku_{xx}, & 0 < x < L, & \quad t > 0 \\ u(0, t) &= a(t) & \text{and} & \quad u(L, t) = b(t), & \quad t > 0 \\ u(x, 0) &= f(x), & 0 \leq x \leq L \end{aligned}$$

where the diffusion constant $k > 0$, and the functions $a(t)$, $b(t)$, and $f(x)$ are C^2 on their respective intervals. Let $T > 0$ be a fixed time and let

$$\alpha = \min_{0 \leq t \leq T} \{a(t)\}, \quad \beta = \min_{0 \leq t \leq T} \{b(t)\}, \quad \text{and} \quad \gamma = \min_{0 \leq x \leq L} \{f(x)\}.$$

If $\mu = \min\{\alpha, \beta, \gamma\}$ and if $u(x, t)$ is any C^2 solution of the initial boundary value problem, then $u(x, t) \geq \mu$ for all $0 \leq x \leq L$ and $0 \leq t \leq T$.

Proof

- ▶ Let $U(x, t) = u(x, t) - v(x, t)$, then

$$U_t = u_t - v_t = ku_{xx} - kv_{xx} = kU_{xx}.$$
- ▶ For $x = 0$, $|U(0, t)| = |a_1(t) - a_2(t)| \leq \epsilon$ by assumption and thus

$$-\epsilon \leq U(0, t) \leq \epsilon \text{ for } 0 \leq t \leq T.$$
- Likewise $-\epsilon \leq U(L, t) \leq \epsilon$ for $0 \leq t \leq T$.
- ▶ For $t = 0$, $|U(x, 0)| = |f_1(x) - f_2(x)| \leq \epsilon$ by assumption and thus

$$-\epsilon \leq U(x, 0) \leq \epsilon \text{ for } 0 \leq x \leq L.$$
- ▶ Applying the Maximum and the Minimum Principles yields

$$-\epsilon \leq U(x, t) \leq \epsilon \text{ or}$$

$$|u(x, t) - v(x, t)| \leq \epsilon$$

for $0 \leq x \leq L$ and $0 \leq t \leq T$.

Continuous Dependence on BC and IC

Theorem

Consider the two initial boundary value problems

$$\begin{array}{llll} u_t &= & ku_{xx} & v_t &= & kv_{xx} \\ u(0, t) &= & a_1(t) & v(0, t) &= & a_2(t) \\ u(L, t) &= & b_1(t) & v(L, t) &= & b_2(t) \\ u(x, 0) &= & f_1(x) & v(x, 0) &= & f_2(x) \end{array}$$

defined for $0 \leq x \leq L$ and $t \geq 0$. Let $T > 0$ and suppose there exists $\epsilon \geq 0$ such that

$$\begin{aligned} |f_1(x) - f_2(x)| &\leq \epsilon \text{ for } 0 \leq x \leq L, \\ |a_1(t) - a_2(t)| &\leq \epsilon \text{ for } 0 \leq t \leq T, \text{ and} \\ |b_1(t) - b_2(t)| &\leq \epsilon \text{ for } 0 \leq t \leq T. \end{aligned}$$

If $u(x, t)$ and $v(x, t)$ are C^2 solutions respectively to the two initial boundary value problems, then for all $0 \leq x \leq L$ and $0 \leq t \leq T$,

$$|u(x, t) - v(x, t)| \leq \epsilon.$$

Uniqueness of Solutions

Corollary

Consider the initial boundary value problem

$$\begin{aligned} u_t &= ku_{xx} + g(x, t), & 0 < x < L, & \quad t > 0 \\ u(0, t) &= a(t) & \text{and} & \quad u(L, t) = b(t), & \quad 0 < x < L \\ u(x, 0) &= f(x), & 0 \leq x \leq L \end{aligned}$$

where the diffusion constant $k > 0$, the function $g(x, t)$ is continuous, and the functions $a(t)$, $b(t)$, and $f(x)$ are C^2 on their respective intervals. If there exists a C^2 solution to this initial boundary value problem then it is unique.

Proof

- ▶ For the purposes of contradiction, suppose there are two solutions $u(x, t)$ and $v(x, t)$.

- ▶ Define $U(x, t) = u(x, t) - v(x, t)$, then

$$\begin{aligned}U_t &= u_t - v_t \\&= ku_{xx} + g(x, t) - (kv_{xx} + g(x, t)) \\&= k(u_{xx} - v_{xx})\end{aligned}$$

$$U_t = kU_{xx} \text{ for } 0 < x < L \text{ and } t > 0.$$

- ▶ $U(0, t) = u(0, t) - v(0, t) = a(t) - a(t) = 0$ and $U(L, t) = 0$ and $U(x, 0) = 0$

- ▶ By the Maximum and Minimum Principles $U(x, t) = 0$ for all $0 \leq x \leq L$ and $t \geq 0$ which implies $u(x, t) = v(x, t)$.

Homework

- ▶ Read Section 4.3
- ▶ Exercises: 21, 22, 23