Maximum Principle and Uniqueness of Solutions

Partial Differential Equations

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Objectives

In this lesson we will explore:

- the Maximum Principle for solutions to the heat equation and its justification,
- the dependence of solutions to the heat equation on the initial and boundary conditions, and
- the uniqueness of solutions to the heat equation and its justification.

The Maximum Principle

Theorem

Consider the initial boundary value problem

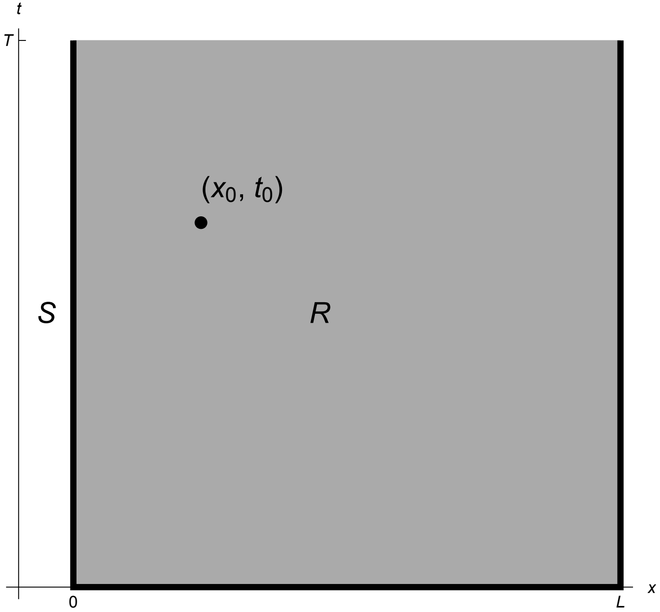
$$\begin{aligned} u_t &= \kappa\, u_{xx}, \ \textit{for } 0 < x < L \textit{ and } t > 0 \\ u(0,t) &= a(t) \textit{ and } u(L,t) = b(t), \ \textit{for } t > 0 \\ u(x,0) &= f(x), \ \textit{for } 0 \le x \le L \end{aligned}$$

where the diffusion constant $\kappa > 0$, and the functions $a(t)$, $b(t)$, and $f(x)$ are $\mathcal{C}^2$ (twice continuously differentiable) on their respective intervals. Let $T > 0$ be a fixed time and let

$$A = \max_{0 \le t \le T} \{a(t)\}, \quad B = \max_{0 \le t \le T} \{b(t)\}, \quad \textit{and} \quad F = \max_{0 \le x \le L} \{f(x)\}.$$

If $M = \max\{A, B, F\}$ and if $u(x,t)$ is any $\mathcal{C}^2$ solution of the initial boundary value problem, then $u(x,t) \le M$ for all $0 \le x \le L$ and $0 \le t \le T$.

Illustration



Example

Consider the IBVP:

$$\begin{aligned} u_t &= 9u_{xx} \text{ for } 0 < x < 3 \text{ and } t > 0 \\ u(0,t) &= 0 = u(3,t) \\ u(x,0) &= 6\sin\frac{\pi x}{3} + 2\sin \pi x \end{aligned}$$

Find an upper bound for the solution.

Solution (1 of 3)

- According to the Maximum Principle, the maximum occurs either where $x = 0$, $x = 3$, or $t = 0$.
- Either the upper bound is $u(0, t) = u(3, t) = 0$ or the maximum occurs where

$$u(x, 0) = f(x) = 6 \sin \frac{\pi x}{3} + 2 \sin \pi x.$$

Solution (1 of 3)

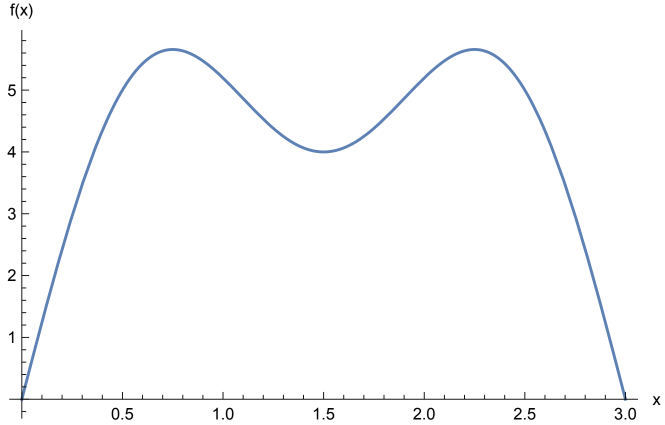
- According to the Maximum Principle, the maximum occurs either where $x = 0$, $x = 3$, or $t = 0$.
- Either the upper bound is $u(0, t) = u(3, t) = 0$ or the maximum occurs where

$$
\begin{aligned}
u(x, 0) = f(x) &= 6 \sin \frac{\pi x}{3} + 2 \sin \pi x. \\
f'(x) &= 2\pi(\cos \frac{\pi x}{3} + \cos \pi x) \\
&= 2\pi \left(\cos \left(\frac{2\pi x}{3} - \frac{\pi x}{3} \right) + \cos \left(\frac{2\pi x}{3} + \frac{\pi x}{3} \right) \right) \\
&= 4\pi \cos \frac{2\pi x}{3} \cos \frac{\pi x}{3}
\end{aligned}
$$

Solution (2 of 3)

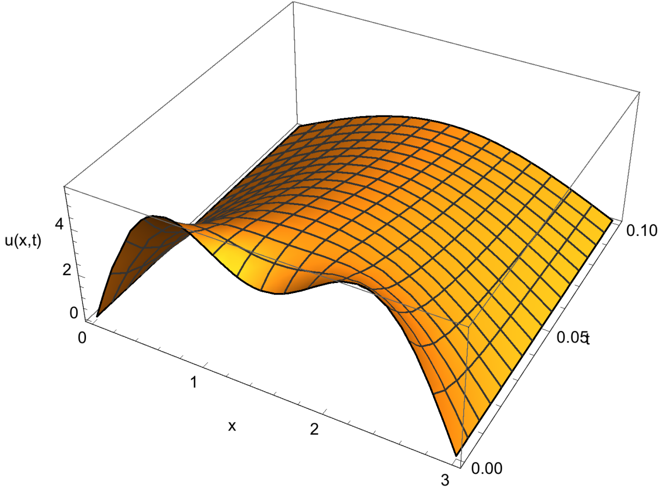
$$f'(x) = 4\pi \cos \frac{2\pi x}{3} \cos \frac{\pi x}{3} = 0$$

has critical numbers $x = 3/4$, $x = 9/4$ (both maxima) and $x = 3/2$ (minimum) in the interval $[0, 3]$.



Solution (3 of 3)

$$u(x,t) \le f(3/4) = f(9/4) = 4\sqrt{2}$$



Minimum Principle

Corollary

Consider the initial boundary value problem

$$
\begin{aligned}
u_t &= \kappa u_{xx}, \text{ for } 0 < x < L \text{ and } t > 0 \\
u(0,t) &= a(t) \text{ and } u(L,t) = b(t), \text{ for } t > 0 \\
u(x,0) &= f(x), \text{ for } 0 \leq x \leq L
\end{aligned}
$$

where the diffusion constant $\kappa > 0$, and the functions $a(t)$, $b(t)$, and $f(x)$ are $\mathcal{C}^2$ on their respective intervals. Let $T > 0$ be a fixed time and let

$$\alpha = \min_{0 \leq t \leq T} \{a(t)\}, \quad \beta = \min_{0 \leq t \leq T} \{b(t)\}, \quad \text{and} \quad \gamma = \min_{0 \leq x \leq L} \{f(x)\}.$$

If $\mu = \min\{\alpha, \beta, \gamma\}$ and if $u(x,t)$ is any $\mathcal{C}^2$ solution of the initial boundary value problem, then $u(x,t) \geq \mu$ for all $0 \leq x \leq L$ and $0 \leq t \leq T$.

Continuous Dependence on BC and IC

Theorem

Consider the two initial boundary value problems

$$\begin{aligned} u_t &= \kappa u_{xx} \\ u(0,t) &= a_1(t) \\ u(L,t) &= b_1(t) \\ u(x,0) &= f_1(x) \end{aligned} \qquad\qquad \begin{aligned} v_t &= \kappa v_{xx} \\ v(0,t) &= a_2(t) \\ v(L,t) &= b_2(t) \\ v(x,0) &= f_2(x) \end{aligned}$$

defined for $0 \leq x \leq L$ *and* $t \geq 0$*. Let* $T > 0$ *and suppose there exists* $\epsilon \geq 0$ *such that*

$$\begin{aligned} |f_1(x) - f_2(x)| &\leq \epsilon \text{ for } 0 \leq x \leq L, \\ |a_1(t) - a_2(t)| &\leq \epsilon \text{ for } 0 \leq t \leq T, \text{ and} \\ |b_1(t) - b_2(t)| &\leq \epsilon \text{ for } 0 \leq t \leq T. \end{aligned}$$

If $u(x,t)$ *and* $v(x,t)$ *are* $\mathcal{C}^2$ *solutions respectively to the two initial boundary value problems, then for all* $0 \leq x \leq L$ *and* $0 \leq t \leq T$*,*

$$|u(x,t) - v(x,t)| \leq \epsilon.$$

Proof

- Let $U(x,t) = u(x,t) - v(x,t)$, then
$$U_t = u_t - v_t = \kappa u_{xx} - \kappa v_{xx} = \kappa U_{xx}.$$
- For $x = 0$, $|U(0,t)| = |a_1(t) - a_2(t)| \leq \epsilon$ by assumption and thus
$$-\epsilon \leq U(0,t) \leq \epsilon \text{ for } 0 \leq t \leq T.$$
Likewise $-\epsilon \leq U(L,t) \leq \epsilon$ for $0 \leq t \leq T$.
- For $t = 0$, $|U(x,0)| = |f_1(x) - f_2(x)| \leq \epsilon$ by assumption and thus
$$-\epsilon \leq U(x,0) \leq \epsilon \text{ for } 0 \leq t \leq L.$$
- Applying the Maximum and the Minimum Principles yields $-\epsilon \leq U(x,t) \leq \epsilon$ or
$$|u(x,t) - v(x,t)| \leq \epsilon$$
for $0 \leq x \leq L$ and $0 \leq t \leq T$.

Uniqueness of Solutions

Corollary

Consider the initial boundary value problem

$$\begin{aligned} u_t &= \kappa u_{xx} + g(x,t), \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= a(t) \text{ and } u(L,t) = b(t), \text{ for } 0 < x < L \\ u(x,0) &= f(x), \text{ for } 0 \le x \le L \end{aligned}$$

where the diffusion constant $\kappa > 0$, the function $g(x,t)$ is continuous, and the functions $a(t)$, $b(t)$, and $f(x)$ are $\mathcal{C}^2$ on their respective intervals. If there exists a $\mathcal{C}^2$ solution to this initial boundary value problem then it is unique.

Proof

- For the purposes of contradiction, suppose there are two solutions $u(x,t)$ and $v(x,t)$.
- Define $U(x,t) = u(x,t) - v(x,t)$, then

$$\begin{aligned}
U_t &= u_t - v_t \\
&= \kappa u_{xx} + g(x,t) - (\kappa v_{xx} + g(x,t)) \\
&= \kappa(u_{xx} - v_{xx}) \\
U_t &= \kappa U_{xx} \text{ for } 0 < x < L \text{ and } t > 0.
\end{aligned}$$

- $U(0,t) = u(0,t) - v(0,t) = a(t) - a(t) = 0$ and $U(L,t) = 0$ and $U(x,0) = 0$
- By the Maximum and Minimum Principles $U(x,t) = 0$ for all $0 \le x \le L$ and $t \ge 0$ which implies $u(x,t) = v(x,t)$.