Energy Integral and Uniqueness of Solutions

MATH 467 *Partial Differential Equations*

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Fall 2022

Objectives

In this lesson we will learn:

- how to express the total energy in a vibrating string as an integral,
- how to use the energy integral to establish the uniqueness of solutions to the wave equation.

Initial Boundary Value Problem

Consider the initial boundary value problem for a string of length $L > 0$.

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

During the derivation of the wave equation we set $c^2 = T_0/\rho$ where ρ is the mass density of the string and T_0 is the tension in the string.

Kinetic Energy

Kinetic energy for a point mass is defined as $mv^2/2$ where v is velocity. For the distributed mass of the string, kinetic energy is

$$K(t) = \frac{1}{2}\int_0^L \rho(u_t(x,t))^2\,dx.$$

Potential Energy

- While at rest, a portion of the string of length Δx is under the tension force T_0.
- When the segment of the string is displaced by $u(x,t)$, its length is approximately $ds = \sqrt{1+(u_x(x,t))^2}\Delta x$.
- Assuming the displacement is small, then

$$ds = \sqrt{1+(u_x(x,t))^2}\Delta x \approx \left(1+\frac{1}{2}(u_x(x,t))^2\right)\Delta x.$$

- The amount by which the length of the segment of the string is changed is

$$\left(1+\frac{1}{2}(u_x(x,t))^2\right)\Delta x - \Delta x = \frac{1}{2}(u_x(x,t))^2\Delta x.$$

- The **potential energy** in the string due to displacement $u(x,t)$ is the work done against force T_0 to stretch the string

$$P(t) = \frac{1}{2}\int_0^L T_0(u_x(x,t))^2\,dx.$$

Total Energy

The **total energy** is the sum of the kinetic and potential energies.

$$E(t) = \frac{1}{2}\int_0^L \left[\rho(u_t(x,t))^2 + T_0(u_x(x,t))^2\right]\,dx$$

Conservation of Energy (1 of 2)

Differentiate the total energy with respect to time.

$$\begin{aligned}
E'(t) &= \frac{1}{2}\int_0^L \left[2\rho u_t(x,t)u_{tt}(x,t) + 2T_0u_x(x,t)u_{xt}(x,t)\right]\,dx \\
&= \int_0^L \left[c^2\rho u_t(x,t)u_{xx}(x,t) + T_0u_x(x,t)u_{xt}(x,t)\right]\,dx \\
&= T_0\int_0^L \left[u_t(x,t)u_{xx}(x,t) + u_x(x,t)u_{xt}(x,t)\right]\,dx \\
&= T_0\int_0^L \frac{d}{dx}\left[u_t(x,t)u_x(x,t)\right]\,dx \\
&= T_0\left[u_t(x,t)u_x(x,t)\right]_{x=0}^{x=L} \\
&= T_0\left[u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t)\right]
\end{aligned}$$

Conservation of Energy (2 of 2)

$$E'(t) = T_0\left[u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t)\right]$$

Since $u(0,t) = u(L,t) = 0$ for all $t \geq 0$ then $u_t(0,t) = u_t(L,t) = 0$. Consequently $E'(t) = 0$ and total energy is conserved.

The total energy for any $t \geq 0$ is the initial energy present in the string.

$$\begin{aligned}
E(t) = E(0) &= \frac{1}{2}\int_0^L \left[\rho(u_t(x,0))^2 + T_0(u_x(x,0))^2\right]\,dx \\
&= \frac{1}{2}\int_0^L \left[\rho(g(x))^2 + T_0(f'(x))^2\right]\,dx
\end{aligned}$$

Uniqueness of Solutions

Theorem
The solution to the initial boundary value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

is unique.

Proof

- Suppose there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ and define $v(x,t) = u_1(x,t) - u_2(x,t)$.
- Function $v(x,t)$ solves the following initial boundary value problem.

$$\begin{aligned} v_{tt} &= c^2 v_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ v(0,t) &= v(L,t) = 0 \\ v(x,0) &= 0 \\ v_t(x,0) &= 0 \end{aligned}$$

- The total energy of solution $v(x,t)$ is 0 which implies $v_t(x,t) = 0$ and $v_x(x,t) = 0$ which implies $v(x,t)$ is constant and therefore $v(x,t) = 0$.

Homework

- Read Section 5.5
- Exercises: 24–26