

Energy Integral and Uniqueness of Solutions

Partial Differential Equations

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Objectives

In this lesson we will learn:

- ▶ how to express the total energy in a vibrating string as an integral,
- ▶ how to use the energy integral to establish the uniqueness of solutions to the wave equation.

Initial Boundary Value Problem

Consider the initial boundary value problem for a string of length $L > 0$.

$$u_{tt} = c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0$$

$$u(0, t) = u(L, t) = 0$$

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = g(x)$$

During the derivation of the wave equation we set $c^2 = T_0/\rho$ where ρ is the mass density of the string and T_0 is the tension in the string.

Kinetic Energy

Kinetic energy for a point mass is defined as $mv^2/2$ where v is velocity. For the distributed mass of the string, kinetic energy is

$$K(t) = \frac{1}{2} \int_0^L \rho (u_t(x, t))^2 dx.$$

Potential Energy

- ▶ While at rest, a portion of the string of length Δx is under the tension force T_0 .
- ▶ When the segment of the string is displaced by $u(x, t)$, its length is approximately $ds = \sqrt{1 + (u_x(x, t))^2} \Delta x$.
- ▶ Assuming the displacement is small, then

$$ds = \sqrt{1 + (u_x(x, t))^2} \Delta x \approx \left(1 + \frac{1}{2}(u_x(x, t))^2\right) \Delta x.$$

- ▶ The amount by which the length of the segment of the string is changed is

$$\left(1 + \frac{1}{2}(u_x(x, t))^2\right) \Delta x - \Delta x = \frac{1}{2}(u_x(x, t))^2 \Delta x.$$

- ▶ The **potential energy** in the string due to displacement $u(x, t)$ is the work done against force T_0 to stretch the string

$$P(t) = \frac{1}{2} \int_0^L T_0 (u_x(x, t))^2 dx.$$

Total Energy

The **total energy** is the sum of the kinetic and potential energies.

$$E(t) = \frac{1}{2} \int_0^L [\rho(u_t(x, t))^2 + T_0(u_x(x, t))^2] dx$$

Conservation of Energy (1 of 2)

Differentiate the total energy with respect to time.

$$\begin{aligned} E'(t) &= \frac{1}{2} \int_0^L [2\rho u_t(x, t) u_{tt}(x, t) + 2T_0 u_x(x, t) u_{xt}(x, t)] \, dx \\ &= \int_0^L [c^2 \rho u_t(x, t) u_{xx}(x, t) + T_0 u_x(x, t) u_{xt}(x, t)] \, dx \\ &= T_0 \int_0^L [u_t(x, t) u_{xx}(x, t) + u_x(x, t) u_{xt}(x, t)] \, dx \\ &= T_0 \int_0^L \frac{d}{dx} [u_t(x, t) u_x(x, t)] \, dx \\ &= T_0 [u_t(x, t) u_x(x, t)]_{x=0}^{x=L} \\ &= T_0 [u_t(L, t) u_x(L, t) - u_t(0, t) u_x(0, t)] \end{aligned}$$

Conservation of Energy (2 of 2)

$$E'(t) = T_0 [u_t(L, t)u_x(L, t) - u_t(0, t)u_x(0, t)]$$

Since $u(0, t) = u(L, t) = 0$ for all $t \geq 0$ then $u_t(0, t) = u_t(L, t) = 0$.
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The total energy for any $t \geq 0$ is the initial energy present in the string.

$$\begin{aligned} E(t) &= E(0) = \frac{1}{2} \int_0^L [\rho(u_t(x, 0))^2 + T_0(u_x(x, 0))^2] dx \\ &= \frac{1}{2} \int_0^L [\rho(g(x))^2 + T_0(f'(x))^2] dx \end{aligned}$$

Uniqueness of Solutions

Theorem

The solution to the initial boundary value problem:

$$u_{tt} = c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0$$

$$u(0, t) = u(L, t) = 0$$

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = g(x)$$

is unique.

Proof

- ▶ Suppose there exist two solutions $u_1(x, t)$ and $u_2(x, t)$ and define $v(x, t) = u_1(x, t) - u_2(x, t)$.
- ▶ Function $v(x, t)$ solves the following initial boundary value problem.

$$v_{tt} = c^2 v_{xx} \text{ for } 0 < x < L \text{ and } t > 0$$

$$v(0, t) = v(L, t) = 0$$

$$v(x, 0) = 0$$

$$v_t(x, 0) = 0$$

- ▶ The total energy of solution $v(x, t)$ is 0 which implies $v_t(x, t) = 0$ and $v_x(x, t) = 0$ which implies $v(x, t)$ is constant and therefore $v(x, t) = 0$.