Wave Equation With Homogeneous Boundary Conditions

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn:

- how to solve the wave equation with homogeneous Dirichlet boundary conditions using separation of variables,
- how to solve the wave equation with homogeneous Neumann boundary conditions using separation of variables,
- terms for describing the components of the solution to the wave equation.

Initial Boundary Value Problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

Since the PDE is linear and homogeneous and the boundary conditions are homogeneous and of Dirichlet type, the method of separation of variables and the Principle of Superposition apply.

Separation of Variables

Assume a product solution of the form $u(x,t) = X(x)T(t)$, differentiate and substitute into the wave equation.

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ X(x)T''(t) &= c^2 X''(x)T(t) \\ \frac{T''(t)}{c^2 T(t)} &= \frac{X''(x)}{X(x)} = -\lambda \end{aligned}$$

where λ is a constant.

This implies the boundary value problem for $X(x)$.

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) &= 0 \\ X(L) &= 0 \end{aligned}$$

Eigenvalues and Eigenfunctions

The only non-trivial eigenfunctions are

$$X_n(x) = \sin \frac{n\pi x}{L}$$

corresponding to the eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$ for $n \in \mathbb{N}$.

With these eigenvalues, the implied ODE for function $T_n(t)$ has the form

$$T_n''(t) + \frac{c^2 n^2 \pi^2}{L^2} T_n(t) = 0$$

and consequently solution

$$T_n(t) = a_n \cos \frac{cn\pi t}{L} + b_n \sin \frac{cn\pi t}{L}.$$

Product Solutions

Functions of the form

$$u_n(x,t) = X_n(x) T_n(t) = \left(a_n \cos \frac{cn\pi t}{L} + b_n \sin \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L}$$

for $n \in \mathbb{N}$ solve the wave equation and satisfy the homogeneous Dirichlet boundary conditions. These solutions are called **fundamental solutions**.

By the Principle of Superposition a sum of fundamental solutions will also solve the wave equation and satisfy the homogeneous Dirichlet boundary conditions.

$$u(x,t) = \sum_{n=1}^{N} \left(a_n \cos \frac{cn\pi t}{L} + b_n \sin \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L}$$

Fundamental Solutions

$$u_n(x,t) = \left(a_n \cos \frac{cn\pi t}{L} + b_n \sin \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L}$$

- This solution is known as the ***n*th harmonic.**
- Solution is periodic in t with period $\frac{2L}{cn}$.
- The number of oscillations per 2π units of time is called the **natural frequency** and is $\frac{cn\pi}{L}$.
- The number of oscillations per unit time is called the **frequency** and is $\frac{cn}{2L}$.
- The **wavelength** of the solution is $\frac{2L}{n}$.
- The intensity of the solution is given by the **amplitude** $\sqrt{a_n^2 + b_n^2}$.

First Harmonic

$$u_1(x,t) = \left(a_1 \cos \frac{c\pi t}{L} + b_1 \sin \frac{c\pi t}{L} \right) \sin \frac{\pi x}{L}$$

- This solution is known as the **first harmonic** or the **fundamental mode.**
- The number of oscillations per 2π units of time is called the **fundamental frequency** and is $\frac{c\pi}{L}$.
- First harmonic is periodic in t with period $\frac{2L}{c}$.
- The nth harmonic has a frequency which is n times the fundamental frequency.

Initial Displacement

We will assume a Fourier series solution to the IBVP.

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{cn\pi t}{L} + b_n \sin \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L}$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} = f(x)$$

Suppose $f(x)$ can be extended to $\mathbb{R}$ as a $2L$–periodic, odd function, then $f(0) = f(L) = f(-L)$ and

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx,$$

for $n \in \mathbb{N}$.

Initial Velocity

If $u(x,t)$ can be differentiated term by term, then

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} \left(-a_n \sin \frac{cn\pi t}{L} + b_n \cos \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} b_n \sin \frac{n\pi x}{L} = g(x)$$

Again, if $g(x)$ can be extended to $\mathbb{R}$ as a $2L$-periodic, odd function then

$$\frac{cn\pi}{L} b_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

for $n \in \mathbb{N}$.

Example: Plucked String

Find the solution to the following IBVP.

$$u_{tt} = u_{xx} \text{ for } 0 < x < 10 \text{ and } t > 0$$

$$u(0,t) = u(10,t) = 0$$

$$u(x,0) = \begin{cases} 2x/5 & \text{if } 0 \le x \le 5/2, \\ 1 & \text{if } 5/2 < x < 15/2, \\ 4 - 2x/5 & \text{if } 15/2 \le x \le 10 \end{cases}$$

$$u_t(x,0) = 0$$

Solution (1 of 2)

The formal solution can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{10} + b_n \sin \frac{n\pi t}{10} \right) \sin \frac{n\pi x}{10}.$$

Since $u_t(x,0) = 0$ then $b_n = 0$ for all $n \in \mathbb{N}$.

$$a_n = \frac{2}{10} \int_0^{10} u(x,0) \sin \frac{n\pi x}{10}\, dx$$

$$= \frac{8}{n^2\pi^2} \left(\sin \frac{n\pi}{4} + \sin \frac{3n\pi}{4} \right)$$

$$u(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{4} + \sin \frac{3n\pi}{4}}{n^2} \cos \frac{n\pi t}{10} \sin \frac{n\pi x}{10}$$

Solution (2 of 2)

Let $f(x)$ be the odd, 20–periodic extension of $u(x,0)$, then

$$f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{4}+\sin\frac{3n\pi}{4}}{n^2}\sin\frac{n\pi x}{10}$$

$$f(x+t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{4}+\sin\frac{3n\pi}{4}}{n^2}\sin\frac{n\pi(x+t)}{10}$$

$$= \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{4}+\sin\frac{3n\pi}{4}}{n^2}\left[\sin\frac{n\pi x}{10}\cos\frac{n\pi t}{10}+\cos\frac{n\pi x}{10}\sin\frac{n\pi t}{10}\right]$$

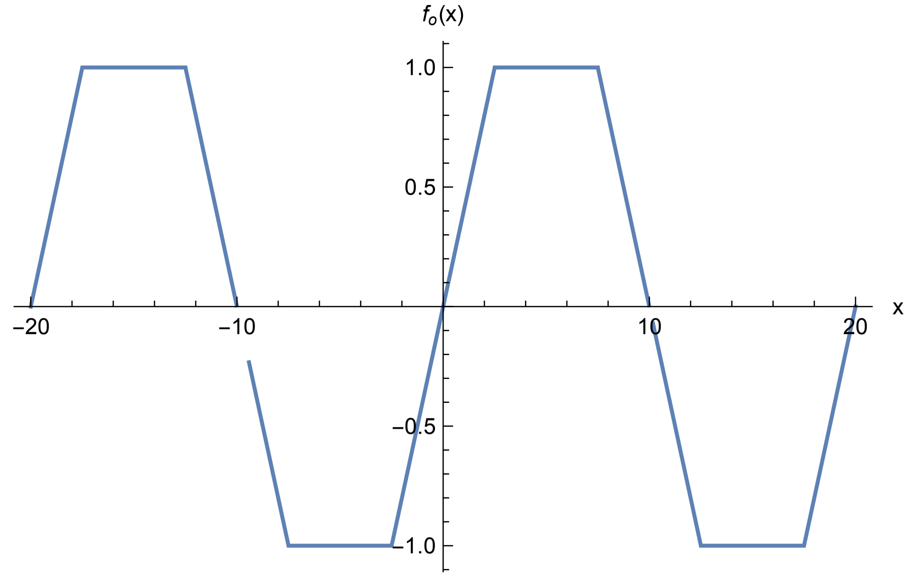
$$f(x-t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{4}+\sin\frac{3n\pi}{4}}{n^2}\sin\frac{n\pi(x-t)}{10}$$

$$= \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{4}+\sin\frac{3n\pi}{4}}{n^2}\left[\sin\frac{n\pi x}{10}\cos\frac{n\pi t}{10}-\cos\frac{n\pi x}{10}\sin\frac{n\pi t}{10}\right]$$

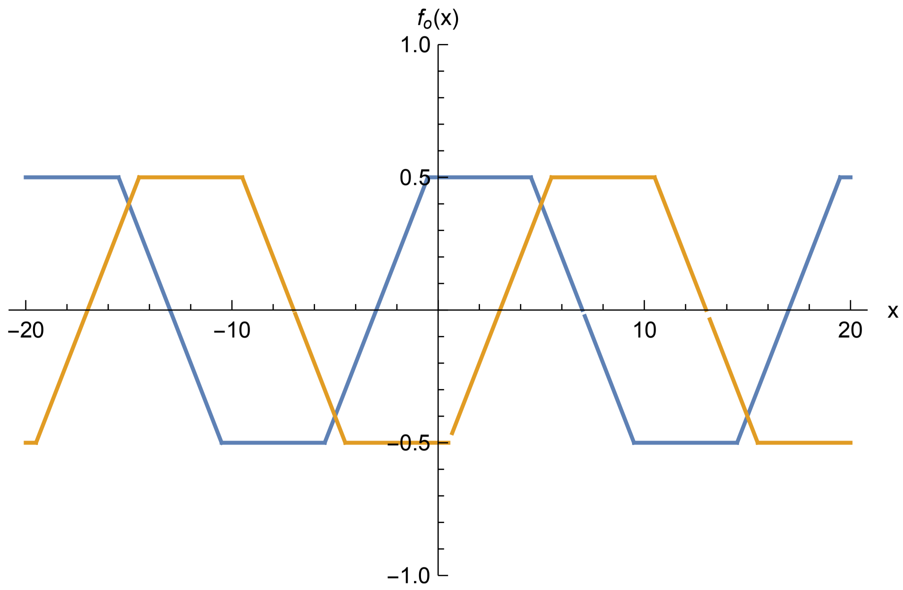
$$f(x+t)+f(x-t) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{4}+\sin\frac{3n\pi}{4}}{n^2}\cos\frac{n\pi t}{10}\sin\frac{n\pi x}{10} = 2u(x,t)$$

$$u(x,t) = \frac{1}{2}\left(f(x+t)+f(x-t)\right)$$

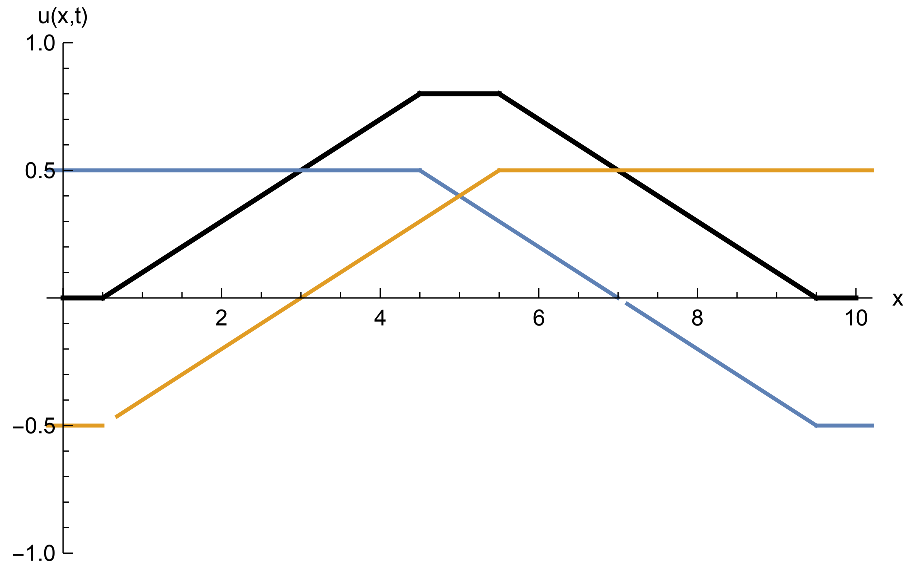
20–Periodic Extension of Initial Displacement



Shifting Initial Displacement Horizontally



Adding Shifts of Initial Displacement



Check by Differentiation

Consider the following IBVP.

$$\begin{aligned} u_{tt} &= u_{xx} \text{ for } 0 < x < 10 \text{ and } t > 0 \\ u(0,t) &= u(10,t) = 0 \\ u(x,0) &= \begin{cases} 2x/5 & \text{if } 0 \le x \le 5/2, \\ 1 & \text{if } 5/2 < x < 15/2, \\ 4 - 2x/5 & \text{if } 15/2 \le x \le 10 \end{cases} \\ u_t(x,0) &= 0 \end{aligned}$$

Show by direct differentiation that

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right)$$

solves the IBVP when f is the odd, 20–periodic extension of $u(x,0)$.

Solution

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(f(x+t) + f(x-t)\right) \\ u_{xx} &= \frac{1}{2}\left(f''(x+t) + f''(x-t)\right) \\ u_{tt} &= \frac{1}{2}\left(f''(x+t) + f''(x-t)\right) \\ u_{tt} &= u_{xx} \end{aligned}$$

Initial displacement:

$$u(x,0) = \frac{1}{2}\left(f(x) + f(x)\right) = f(x)$$

Initial velocity:

$$u_t(x,0) = \frac{1}{2}\left(f'(x) - f'(x)\right) = 0$$

Dirichlet boundary conditions:

$$\begin{aligned} u(0,t) &= \frac{1}{2}(f(t) + f(-t)) = 0 \ (f \text{ is odd}) \\ u(10,t) &= \frac{1}{2}(f(10+t) + f(10-t)) = 0 \ (f \text{ is odd, 20-periodic}) \end{aligned}$$

Example: Struck String

Find the solution to the following IBVP.

$$\begin{aligned} u_{tt} &= u_{xx} \text{ for } 0 < x < 10 \text{ and } t > 0 \\ u(0,t) &= u(10,t) = 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= \begin{cases} x/5 & \text{if } 0 \le x \le 5, \\ 2 - x/5 & \text{if } 5 < x \le 10 \end{cases} \end{aligned}$$

Solution (1 of 4)

The formal solution can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{10} + b_n \sin \frac{n\pi t}{10} \right) \sin \frac{n\pi x}{10}.$$

Since $u(x,0) = 0$ then $a_n = 0$ for all $n \in \mathbb{N}$.

$$\begin{aligned} b_n &= \frac{2}{n\pi} \int_0^{10} u_t(x,0) \sin \frac{n\pi x}{10} \, dx \\ &= \frac{80}{n^3\pi^3} \sin \frac{n\pi}{2} \\ u(x,t) &= \frac{80}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^3} \sin \frac{n\pi t}{10} \sin \frac{n\pi x}{10} \end{aligned}$$

Solution (2 of 4)

Let $g(x)$ be the odd, 20–periodic extension of $u_t(x,0)$, then

$$g(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi x}{10}.$$

Define $G(x) = \int_0^x g(s)\,ds$ and integrate the Fourier series term by term.

$$\begin{aligned}
G(x) &= \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\int_0^x \sin\frac{n\pi s}{10}\,ds\\
&= \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\left[-\frac{10}{n\pi}\cos\frac{n\pi s}{10}\right]_{s=0}^{s=x}\\
&= \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\left(1-\cos\frac{n\pi x}{10}\right)\\
&= \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3} - \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\cos\frac{n\pi x}{10}
\end{aligned}$$

Solution (3 of 4)

$$\begin{aligned}
G(x+t) &= \int_0^{x+t} g(s)\,ds\\
&= \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3} - \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\cos\frac{n\pi(x+t)}{10}\\
&= \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3} - \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\left[\cos\frac{n\pi x}{10}\cos\frac{n\pi t}{10} - \sin\frac{n\pi x}{10}\sin\frac{n\pi t}{10}\right]\\
G(x-t) &= \int_0^{x-t} g(s)\,ds\\
&= \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3} - \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\cos\frac{n\pi(x-t)}{10}\\
&= \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3} - \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\left[\cos\frac{n\pi x}{10}\cos\frac{n\pi t}{10} + \sin\frac{n\pi x}{10}\sin\frac{n\pi t}{10}\right]
\end{aligned}$$

Solution (4 of 4)

$$G(x+t) = \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3} - \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\left[\cos\frac{n\pi x}{10}\cos\frac{n\pi t}{10} - \sin\frac{n\pi x}{10}\sin\frac{n\pi t}{10}\right]$$

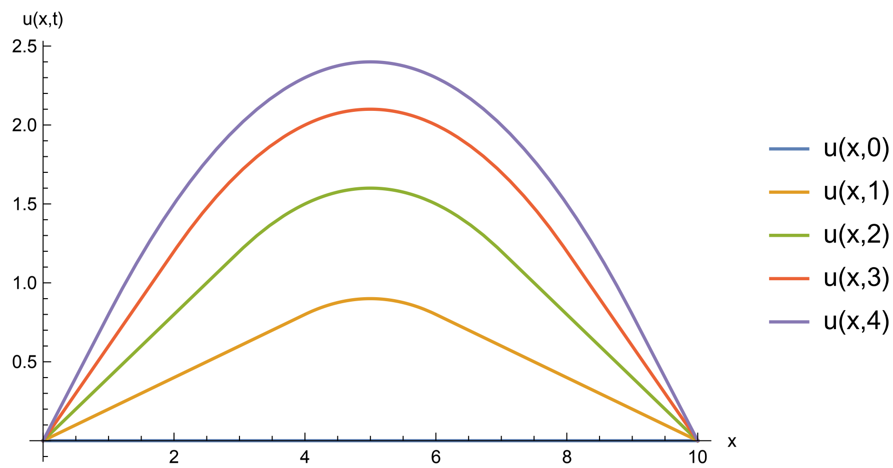
$$G(x-t) = \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3} - \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\left[\cos\frac{n\pi x}{10}\cos\frac{n\pi t}{10} + \sin\frac{n\pi x}{10}\sin\frac{n\pi t}{10}\right]$$

Subtract the two equations.

$$G(x+t) - G(x-t) = \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\left(\sin\frac{n\pi x}{10}\sin\frac{n\pi t}{10} + \sin\frac{n\pi x}{10}\sin\frac{n\pi t}{10}\right)$$

$$\begin{aligned}
\int_0^{x+t} g(s)\,ds - \int_0^{x-t} g(s)\,dx &= \frac{160}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\sin\frac{n\pi t}{10}\sin\frac{n\pi x}{10}\\
\int_{x-t}^{x+t} g(s)\,ds &= \frac{160}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\sin\frac{n\pi t}{10}\sin\frac{n\pi x}{10}\\
\frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds &= \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^3}\sin\frac{n\pi t}{10}\sin\frac{n\pi x}{10} = u(x,t)
\end{aligned}$$

Illustration



Check by Differentiation

Consider the following IBVP.

$$\begin{aligned} u_{tt} &= u_{xx} \text{ for } 0 < x < 10 \text{ and } t > 0 \\ u(0,t) &= u(10,t) = 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= \begin{cases} x/5 & \text{if } 0 \leq x \leq 5, \\ 2 - x/5 & \text{if } 5 < x \leq 10 \end{cases} \end{aligned}$$

Show by direct differentiation that

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

solves the IBVP when g is the odd, 20–periodic extension of $u_t(x,0)$.

Solution (1 of 2)

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\ u_t(x,t) &= \frac{1}{2}(g(x+t) + g(x-t)) \\ u_{tt}(x,t) &= \frac{1}{2}(g'(x+t) - g'(x-t)) \\ u_x(x,t) &= \frac{1}{2}(g(x+t) - g(x-t)) \\ u_{xx}(x,t) &= \frac{1}{2}(g'(x+t) - g'(x-t)) \\ u_{tt} &= u_{xx} \end{aligned}$$

Solution (2 of 2)

- Displacement when $t = 0$,

$$u(x,0) = \frac{1}{2}\int_{x}^{x} g(s)\,ds = 0$$

  if g is continuous at x.
- Velocity when $t = 0$,

$$u_t(x,0) = \frac{1}{2}(g(x) + g(x)) = g(x).$$

- Boundary condition at $x = 0$,

$$u(0,t) = \frac{1}{2}\int_{-t}^{t} g(s)\,ds = 0$$

  since $g(x)$ is odd.
- Boundary condition at $x = 10$,

$$u(10,t) = \frac{1}{2}\int_{10-t}^{10+t} g(s)\,ds = 0$$

  since $g(x)$ is odd, 20-periodic.

Combination

Suppose $u(x,t)$ and $v(x,t)$ solve the respective IBVPs for $0 < x < L$ and $t > 0$:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} & v_{tt} &= c^2 v_{xx} \\ u(0,t) &= u(L,t) = 0 & v(0,t) &= v(L,t) = 0 \\ u(x,0) &= f(x) & v(x,0) &= 0 \\ u_t(x,0) &= 0 & v_t(x,0) &= g(x) \end{aligned}$$

Question: what IBVP would $w(x,t) = u(x,t) + v(x,t)$ solve?

$$\begin{aligned} w_{tt} &= c^2 w_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ w(0,t) &= w(L,t) = 0 \\ w(x,0) &= f(x) \\ w_t(x,0) &= g(x) \end{aligned}$$

Example

Find the solution to the IBVP:

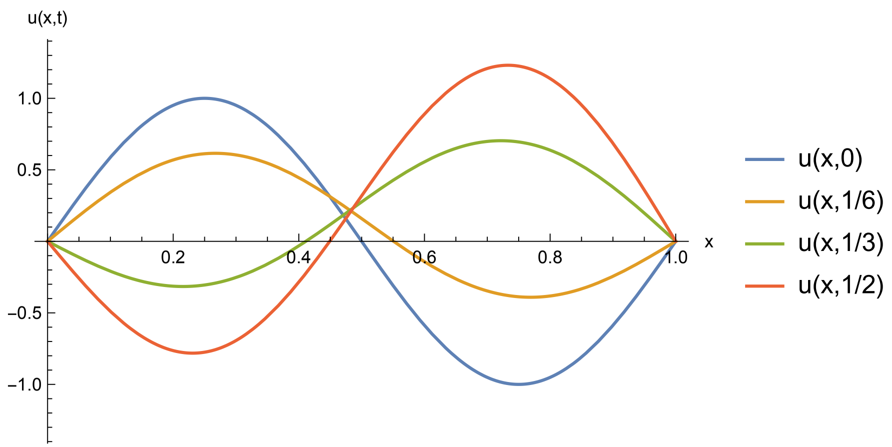
$$\begin{aligned} u_{tt} &= u_{xx} \text{ for } 0 < x < 1 \text{ and } t > 0 \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= \sin(2\pi x) \\ u_t(x,0) &= \sin(\pi x) \end{aligned}$$

Solution

Let $f(x) = \sin(2\pi x)$ and $g(x) = \sin(\pi x)$ which both odd functions and 2–periodic.

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\sin(2\pi(x+t)) + \sin(2\pi(x-t))\right) + \frac{1}{2}\int_{x-t}^{x+t} \sin(\pi s)\, ds \\ &= \sin(2\pi x)\cos(2\pi t) + \frac{1}{\pi}\sin(\pi x)\sin(\pi t) \end{aligned}$$

Graph



Homework

- Read Section 5.1
- Exercises: 1–5