

The Nonhomogeneous Wave Equation

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn:

- ▶ a decomposition approach to solving nonhomogeneous wave equations.

General Nonhomogeneous Wave Equation

Consider the following initial boundary value problem:

$$\begin{aligned}u_{tt} &= c^2 u_{xx} + F(x, t) \text{ for } 0 < x < L \text{ and } t > 0 \\u(0, t) &= \phi(t) \text{ and } u(L, t) = \psi(t) \text{ for } t > 0 \\u(x, 0) &= f(x) \text{ and } u_t(x, 0) = g(x) \text{ for } 0 < x < L.\end{aligned}$$

The solution to the IBVP can be found by solving two simpler initial boundary value problems and using the Principle of Superposition to reconstruct the full solution.

Two Sub-Problems

$$\begin{aligned}v_{tt} &= c^2 v_{xx} + F(x, t) \text{ for } 0 < x < L \text{ and } t > 0 \\v(0, t) &= v(L, t) = 0 \text{ for } t > 0 \\v(x, 0) &= v_t(x, 0) = 0 \text{ for } 0 < x < L\end{aligned}$$

The IBVP above contains the nonhomogeneous PDE.

$$\begin{aligned}w_{tt} &= c^2 w_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\w(0, t) &= \phi(t) \text{ and } w(L, t) = \psi(t) \text{ for } t > 0 \\w(x, 0) &= f(x) \text{ and } w_t(x, 0) = g(x) \text{ for } 0 < x < L.\end{aligned}$$

The IBVP above contains the nonhomogeneous BCs.

If $v(x, t)$ and $w(x, t)$ solve their respective IBVPs, then $u(x, t) = v(x, t) + w(x, t)$ solves the original IBVP.

Homogeneous PDE with Nonhomogeneous BCs

Consider the IBVP with nonhomogeneous BCs:

$$\begin{aligned}w_{tt} &= c^2 w_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\w(0, t) &= \phi(t) \text{ for } t > 0 \\w(L, t) &= \psi(t) \text{ for } t > 0 \\w(x, 0) &= f(x) \text{ for } 0 < x < L \\w_t(x, 0) &= g(x) \text{ for } 0 < x < L.\end{aligned}$$

Assume the solution can be written as $w(x, t) = y(x, t) + r(x, t)$ where $r(x, t)$ is a **reference function** satisfying the nonhomogeneous BCs and $y(x, t)$ is an unknown function, to be found later.

Reference Function

Find any function $r(x, t)$ which satisfies the nonhomogeneous BCs:

$$\begin{aligned}w(0, t) &= \phi(t) \text{ for } t > 0 \\w(L, t) &= \psi(t) \text{ for } t > 0.\end{aligned}$$

Many solutions are possible, but a straightforward one is

$$r(x, t) = \frac{x}{L} [\psi(t) - \phi(t)] + \phi(t).$$

If $w(x, t) = y(x, t) + r(x, t)$ solves the IBVP given earlier with nonhomogeneous BCs, find the IBVP which $y(x, t)$ solves.

Related IBVP with Homogeneous BCs

$$w_{tt} = y_{tt} + \frac{x}{L} [\psi''(t) - \phi''(t)] + \phi''(t)$$

$$w_{xx} = y_{xx}$$

$$y_{tt} = c^2 y_{xx} - \frac{x}{L} [\psi''(t) - \phi''(t)] - \phi''(t) \text{ for } 0 < x < L, t > 0$$

$$y(0, t) = y(L, t) = 0 \text{ for } t > 0$$

$$y(x, 0) = f(x) - \frac{x}{L} [\psi(0) - \phi(0)] - \phi(0) \text{ for } 0 < x < L$$

$$y_t(x, 0) = g(x) - \frac{x}{L} [\psi'(0) - \phi'(0)] - \phi'(0) \text{ for } 0 < x < L.$$

Remark: the IBVP for $y(x, t)$ has homogeneous Dirichlet boundary conditions, but a nonhomogeneous PDE. We must decompose it into two additional sub-problems in order to solve it.

Two More Sub-Problems

Note: the dependent variables are again named v and w though they are not the same functions as mentioned earlier.

IBVP with homogeneous PDE:

$$w_{tt} = c^2 w_{xx} \text{ for } 0 < x < L, t > 0$$

$$w(0, t) = w(L, t) = 0 \text{ for } t > 0$$

$$w(x, 0) = f(x) - \frac{x}{L} [\psi(0) - \phi(0)] - \phi(0) \text{ for } 0 < x < L$$

$$w_t(x, 0) = g(x) - \frac{x}{L} [\psi'(0) - \phi'(0)] - \phi'(0) \text{ for } 0 < x < L.$$

IBVP with nonhomogeneous PDE:

$$v_{tt} = c^2 v_{xx} - \frac{x}{L} [\psi''(t) - \phi''(t)] - \phi''(t) \text{ for } 0 < x < L, t > 0$$

$$v(0, t) = v(L, t) = 0 \text{ for } t > 0$$

$$v(x, 0) = 0 \text{ for } 0 < x < L$$

$$v_t(x, 0) = 0 \text{ for } 0 < x < L.$$

IBVP With Homogeneous PDE

$$\begin{aligned}w_{tt} &= c^2 w_{xx} \text{ for } 0 < x < L, t > 0 \\w(0, t) &= w(L, t) = 0 \text{ for } t > 0 \\w(x, 0) &= f(x) - \frac{x}{L} [\psi(0) - \phi(0)] - \phi(0) \text{ for } 0 < x < L \\w_t(x, 0) &= g(x) - \frac{x}{L} [\psi'(0) - \phi'(0)] - \phi'(0) \text{ for } 0 < x < L.\end{aligned}$$

Remark: this IBVP has homogeneous Dirichlet BCs and thus we can express the solution as a Fourier series:

$$w(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{cn\pi t}{L} + b_n \sin \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L}.$$

IBVP with nonhomogeneous PDE

$$\begin{aligned}v_{tt} &= c^2 v_{xx} - \frac{x}{L} [\psi''(t) - \phi''(t)] - \phi''(t) \text{ for } 0 < x < L, t > 0 \\v(0, t) &= v(L, t) = 0 \text{ for } t > 0 \\v(x, 0) &= 0 \text{ for } 0 < x < L \\v_t(x, 0) &= 0 \text{ for } 0 < x < L.\end{aligned}$$

Remark: this IBVP has homogeneous Dirichlet BCs and ICs of zero. We will solve this nonhomogeneous PDE in more generality by assuming the nonhomogeneity is a function $F(x, t)$.

Finding the Solution to the Nonhomogeneous PDE

$$\begin{aligned}u_{tt} &= c^2 u_{xx} + F(x, t) \text{ for } 0 < x < L, t > 0 \\u(0, t) &= u(L, t) = 0 \text{ for } t > 0 \\u(x, 0) &= 0 \text{ for } 0 < x < L \\u_t(x, 0) &= 0 \text{ for } 0 < x < L.\end{aligned}$$

Assume the solution has the form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L}.$$

Differentiate and substitute into the PDE.

Differentiating the Solution

$$\begin{aligned}u_{tt} &= c^2 u_{xx} + F(x, t) \\ \sum_{n=1}^{\infty} T_n''(t) \sin \frac{n\pi x}{L} &= -c^2 \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 T_n(t) \sin \frac{n\pi x}{L} + F(x, t)\end{aligned}$$

Rearrange terms to isolate the nonhomogeneous term.

$$\sum_{n=1}^{\infty} \left[T_n''(t) + \left(\frac{n\pi c}{L}\right)^2 T_n(t) \right] \sin \frac{n\pi x}{L} = F(x, t)$$

Multiply both sides by $\sin(m\pi x)/L$ and integrate from $x = 0$ to $x = L$.

Orthogonality

Since $\sin(n\pi x)/L$ and $\sin(m\pi x)/L$ are orthogonal when $n \neq m$ then

$$T_m''(t) + \left(\frac{m\pi c}{L}\right)^2 T_m(t) = \frac{2}{L} \int_0^L F(x, t) \sin \frac{m\pi x}{L} dx = F_m(t)$$

for $m \in \mathbb{N}$.

For $n = 1, 2, \dots$ we must solve the initial value problems:

$$\begin{aligned} T_n''(t) + \left(\frac{n\pi c}{L}\right)^2 T_n(t) &= F_n(t) \\ T_n(0) &= 0 \\ T_n'(0) &= 0. \end{aligned}$$

Example

Find the solution to the following IBVP:

$$\begin{aligned} u_{tt} &= u_{xx} + t \sin x \text{ for } 0 < x < \pi \text{ and } t > 0 \\ u(0, t) &= u(\pi, t) = 0 \text{ for } t > 0 \\ u(x, 0) &= \sin x \text{ for } 0 < x < \pi \\ u_t(x, 0) &= \sin(3x) \text{ for } 0 < x < \pi. \end{aligned}$$

Solution (1 of 7)

Find the solution to the homogeneous IBVP:

$$\begin{aligned}w_{tt} &= w_{xx} \text{ for } 0 < x < \pi \text{ and } t > 0 \\w(0, t) &= w(\pi, t) = 0 \text{ for } t > 0 \\w(x, 0) &= \sin x \text{ for } 0 < x < \pi \\w_t(x, 0) &= \sin(3x) \text{ for } 0 < x < \pi.\end{aligned}$$

This can be found readily in d'Alembertian form.

Solution (2 of 7)

$$\begin{aligned}w(x, t) &= \frac{1}{2} (\sin(x - t) + \sin(x + t)) + \frac{1}{2} \int_{x-t}^{x+t} \sin(3s) \, ds \\&= \frac{1}{2} (\sin(x - t) + \sin(x + t)) + \frac{1}{6} (\cos 3(x - t) - \cos 3(x + t)) \\&= \sin x \cos t + \frac{1}{3} \sin(3x) \sin(3t)\end{aligned}$$

Solution (3 of 7)

Now find the solution to the IBVP containing the nonhomogeneous PDE and zero initial conditions.

$$\begin{aligned}v_{tt} &= v_{xx} + t \sin x \text{ for } 0 < x < \pi \text{ and } t > 0 \\v(0, t) &= v(\pi, t) = 0 \text{ for } t > 0 \\v(x, 0) &= 0 \text{ for } 0 < x < \pi \\v_t(x, 0) &= 0 \text{ for } 0 < x < \pi.\end{aligned}$$

Make the assumption that the solution can be expressed as

$$v(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin(nx).$$

Solution (4 of 7)

Differentiating $v(x, t)$ and substituting into the PDE produce:

$$\begin{aligned}\sum_{n=1}^{\infty} T_n''(t) \sin(nx) &= - \sum_{n=1}^{\infty} n^2 T_n(t) \sin(nx) + t \sin x \\ \sum_{n=1}^{\infty} (T_n''(t) + n^2 T_n(t)) \sin(nx) &= t \sin x.\end{aligned}$$

Multiply both sides by $\sin(mx)$ and integrate over $[0, \pi]$.

Solution (5 of 7)

$$\begin{aligned}T_m''(t) + m^2 T_m(t) &= \frac{2}{\pi} \int_0^\pi t \sin x \sin(mx) dx \\&= \begin{cases} t & \text{if } m = 1 \\ 0 & \text{if } m = 2, 3, \dots \end{cases}\end{aligned}$$

Solve the initial value problems:

$$\begin{aligned}T_1''(t) + T_1(t) &= t \\T_n''(t) + n^2 T_n(t) &= 0\end{aligned}$$

for $n = 2, 3, \dots$ with zero initial conditions.

Solution (6 of 7)

We can immediately check that $T_n(t) = 0$ for $n = 2, 3, \dots$

$$T_1(t) = A_1 \cos t + B_1 \sin t + t$$

Making use of the initial conditions ($T_1(0) = T_1'(0) = 0$) reveals,

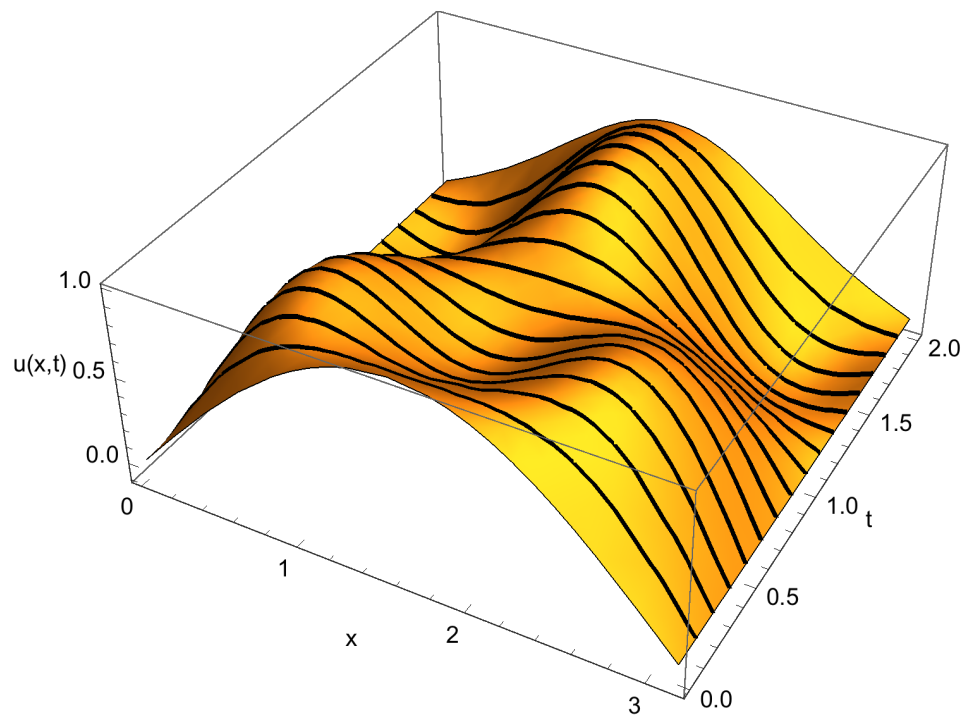
$$T_1(t) = t - \sin t.$$

This implies,

$$v(x, t) = T_1(t) \sin x = (t - \sin t) \sin x.$$

Solution (7 of 7)

$$\begin{aligned} u(x, t) &= w(x, t) + v(x, t) \\ &= \sin x \cos t + \frac{1}{3} \sin(3x) \sin(3t) + (t - \sin t) \sin x. \end{aligned}$$



Homework

- Read Sections 5.4
- Exercises: 18–20