D'Alembert's Solution to the Wave Equation

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn:

- a change of variable technique which simplifies the wave equation,
- d'Alembert's solution to the wave equation which avoids the summing of a Fourier series solution.

Wave Equation

Consider the initial value problem for the unbounded, homogeneous one-dimensional wave equation

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\
u(x,0) &= f(x) \\
u_t(x,0) &= g(x).
\end{aligned}
$$

Rewrite the PDE by making the change of variables

$$
\begin{aligned}
\xi &= x + c\,t \\
\eta &= x - c\,t.
\end{aligned}
$$

Change of Variables

First partial derivatives:

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$$
$$u_t = u_\xi \xi_t + u_\eta \eta_t = c(u_\xi - u_\eta)$$

Second partial derivatives:

$$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$
$$u_{tt} = c(u_{\xi\xi}\xi_t + u_{\xi\eta}\eta_t) - c(u_{\eta\xi}\xi_t + u_{\eta\eta}\eta_t) = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$

Substitution into Wave Equation

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} \\
c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) &= c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \\
u_{\xi\eta} &= 0
\end{aligned}
$$

Integrate both sides of the equation.

$$
\begin{aligned}
u_{\xi} &= \phi(\xi) \\
u(\xi, \eta) &= \int \phi(\xi)\, d\xi \\
&= \Phi(\xi) + \Psi(\eta)
\end{aligned}
$$

Functions Φ and Ψ are arbitrary smooth functions.

Return to the Original Variables

$$u(\xi, \eta) = \Phi(\xi) + \Psi(\eta)$$
$$u(x, t) = \Phi(x + c\,t) + \Psi(x - c\,t)$$

This is referred to as **d'Alembert's general solution to the wave equation**.

Question: can Φ and Ψ be chosen to satisfy the initial conditions?

Plucked String (1 of 2)

$$
\begin{aligned}
u(x,t) &= \Phi(x + c\,t) + \Psi(x - c\,t) \\
u(x,0) &= \Phi(x) + \Psi(x) = f(x) \\
u_t(x,0) &= c\,\Phi'(x) - c\,\Psi'(x) = 0 \\
0 &= \Phi'(x) - \Psi'(x)
\end{aligned}
$$

Integrate the last equation.

$$
\Phi(x) = \Psi(x) + K
$$

where K is an arbitrary constant.

Substituting into the equation for the initial displacement produces

$$
\begin{aligned}
2\Psi(x) + K &= f(x) \\
\Psi(x) &= \frac{1}{2}\left(f(x) - K\right) \\
\Phi(x) &= \frac{1}{2}\left(f(x) + K\right).
\end{aligned}
$$

Plucked String (2 of 2)

Consequently if f is twice differentiable, then

$$u(x,t) = \frac{1}{2}\left[f(x + c\,t) + f(x - c\,t)\right]$$

solves the initial value problem describing the plucked string.

Struck String (1 of 2)

$$
\begin{aligned}
u(x,t) &= \Phi(x + c\,t) + \Psi(x - c\,t) \\
u(x,0) &= \Phi(x) + \Psi(x) = 0 \\
u_t(x,0) &= c\,\Phi'(x) - c\,\Psi'(x) = g(x)
\end{aligned}
$$

Differentiating the 2nd equation reveals $\Phi'(x) = -\Psi'(x)$

Substituting into the 3rd equation produces

$$
\begin{aligned}
2c\,\Phi'(x) &= g(x) \\
\Phi(x) &= \frac{1}{2c}\int_0^x g(s)\,ds + K \\
\Psi(x) &= -\frac{1}{2c}\int_0^x g(s)\,ds - K.
\end{aligned}
$$

Struck String (2 of 2)

Consequently if g is continuously differentiable, then

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\left[\int_0^{x+c\,t} g(s)\,ds - \int_0^{x-c\,t} g(s)\,ds\right] \\ &= \frac{1}{2c}\left[\int_0^{x+c\,t} g(s)\,ds + \int_{x-c\,t}^{0} g(s)\,ds\right] \\ &= \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} g(s)\,ds \end{aligned}$$

solves the initial value problem describing the struck string.

Nonzero Displacement and Velocity

By the Principle of Superposition, the general solution is

$$u(x,t) = \frac{1}{2}\left[f(x - c\,t) + f(x + c\,t)\right] + \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} g(s)\,ds.$$

Example: Plucked String

Determine the solution to the initial value problem:

$$u_{tt} = c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0$$

$$u(x,0) = f(x) = \begin{cases} 2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$u_t(x,0) = 0$$

Solution (1 of 7)

Using d'Alembert's solution

$$u(x, t) = \frac{1}{2}\left[f(x + c\,t) + f(x - c\,t)\right].$$

Note:

- Along lines where $x + c\,t$ is constant the term $f(x + c\,t)$ is constant.
- Likewise along lines where $x - c\,t$ is constant the term $f(x - c\,t)$ is constant.
- These lines are called **characteristics**.

Solution (2 of 7)

$$f(x) = \begin{cases} 2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

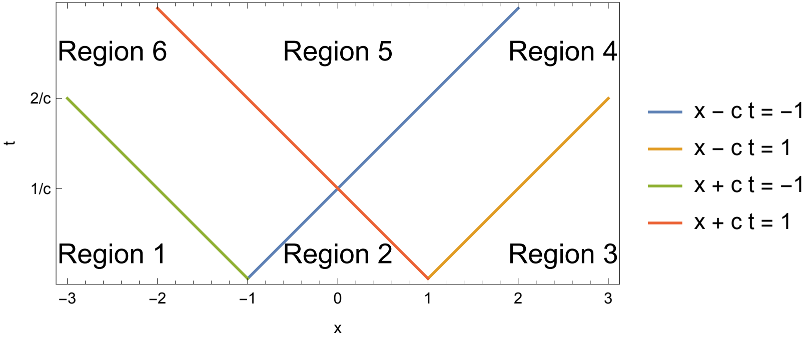
$$f(x + c\,t) = \begin{cases} 2 & \text{if } -1 < x + c\,t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{2} f(x + c\,t) = \begin{cases} 1 & \text{if } -1 - c\,t < x < 1 - c\,t \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{2} f(x - c\,t) = \begin{cases} 1 & \text{if } -1 + c\,t < x < 1 + c\,t \\ 0 & \text{otherwise} \end{cases}$$

Remark: the characteristics where $x + c\,t = \pm 1$ and $x - c\,t = \pm 1$ help determine the solution.

Solution (3 of 7)



Solution (4 of 7)

Region 1: $\{(x,t) \,|\, x + c\,t < -1\}$

Region 2: $\{(x,t) \,|\, -1 < x - c\,t \text{ and } x + c\,t < 1\}$

Region 3: $\{(x,t) \,|\, 1 < x - c\,t\}$

Region 4: $\{(x,t) \,|\, 1 < x + c\,t \text{ and } -1 < x - c\,t < 1\}$

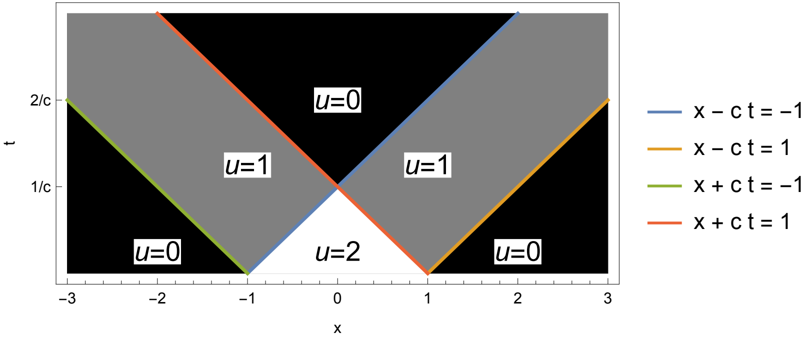
Region 5: $\{(x,t) \,|\, 1 < x + c\,t \text{ and } x - c\,t < -1\}$

Region 6: $\{(x,t) \,|\, -1 < x + c\,t < 1 \text{ and } x - c\,t < -1\}$

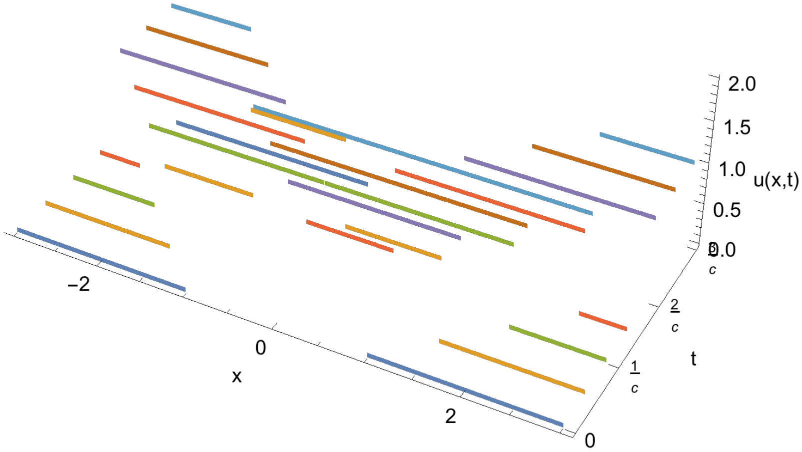
Solution (5 of 7)

$$
\begin{aligned}
u(x,t) &= \frac{1}{2} f(x + c\,t) + \frac{1}{2} f(x - c\,t) \\
&= \begin{cases}
0 & \text{if } x + c\,t < -1 \\
2 & \text{if } -1 < x - c\,t < 1 \text{ and } -1 < x + c\,t < 1 \\
0 & \text{if } 1 < x - c\,t \\
1 & \text{if } 1 < x + c\,t \text{ and } -1 < x - c\,t < 1 \\
0 & \text{if } 1 < x + c\,t \text{ and } x - c\,t < -1 \\
1 & \text{if } -1 < x + c\,t < 1 \text{ and } x - c\,t < -1
\end{cases}
\end{aligned}
$$

Solution (6 of 7)



Solution (7 of 7)



Example: Struck String

Determine the solution to the initial value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= g(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Solution (1 of 6)

Define the function

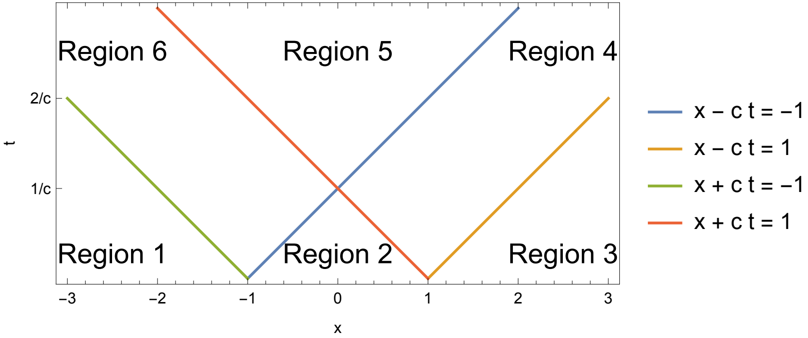
$$G(z) = \int_0^z g(w)\,dw = \begin{cases} -1 & \text{if } z < -1 \\ z & \text{if } -1 \le z \le 1 \\ 1 & \text{if } z > 1. \end{cases}$$

then

$$u(x,t) = \frac{1}{2c}\left[G(x + c\,t) - G(x - c\,t)\right].$$

As in the previous example, the characteristics $x + c\,t = \pm 1$ and $x - c\,t = \pm 1$ divide the xt-plane into six regions.

Solution (2 of 6)



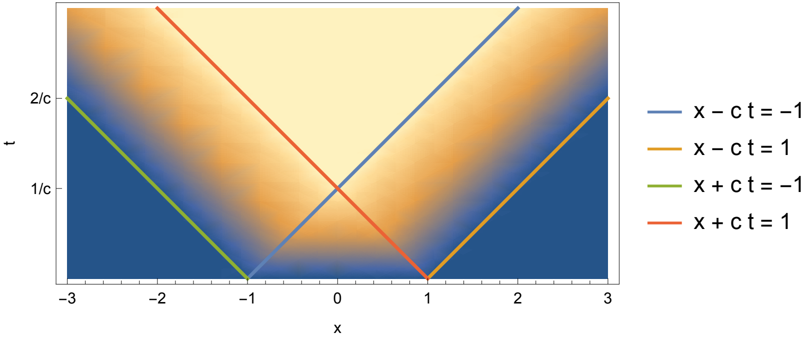
Solution (3 of 6)

Region 1: $\{(x,t)\,|\,x+c\,t<-1\}$
Region 2: $\{(x,t)\,|\,-1<x-c\,t \text{ and } x+c\,t<1\}$
Region 3: $\{(x,t)\,|\,1<x-c\,t\}$
Region 4: $\{(x,t)\,|\,1<x+c\,t \text{ and } -1<x-c\,t<1\}$
Region 5: $\{(x,t)\,|\,1<x+c\,t \text{ and } x-c\,t<-1\}$
Region 6: $\{(x,t)\,|\,-1<x+c\,t<1 \text{ and } x-c\,t<-1\}$

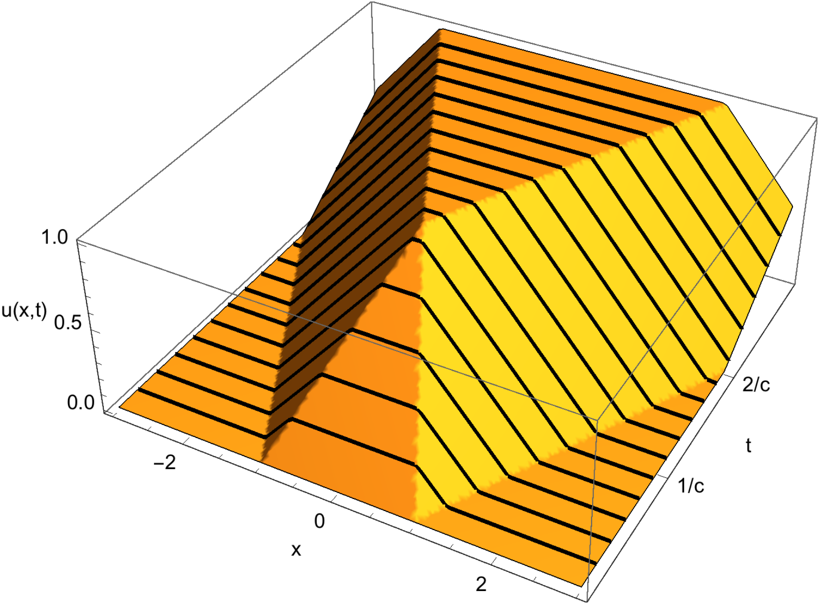
Solution (4 of 6)

$$
\begin{aligned}
u(x,t) &= \frac{1}{2c} G(x+c\,t) - \frac{1}{2c} G(x-c\,t) \\
&= \frac{1}{2c} \left\{ \begin{array}{rl}
0 & \text{if } x+c\,t < -1 \\
2c\,t & \text{if } -1 < x-c\,t \text{ and } x+c\,t < 1 \\
0 & \text{if } 1 < x-c\,t \\
1-x+c\,t & \text{if } 1 < x+c\,t \text{ and } -1 < x-c\,t < 1 \\
2 & \text{if } 1 < x+c\,t \text{ and } x-c\,t < -1 \\
1+x+c\,t & \text{if } x-c\,t < -1 \text{ and } -1 < x+c\,t < 1
\end{array} \right.
\end{aligned}
$$

Solution (5 of 6)



Solution (6 of 6)



Domain of Dependence (1 of 2)

In general the solution to the initial value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

can be expressed as

$$u(x,t) = \frac{1}{2}\left[f(x+c\,t) + f(x-c\,t)\right] + \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} g(s)\,ds.$$

At the point (x_0, t_0) then

$$u(x_0,t_0) = \frac{1}{2}\left[f(x_0+c\,t_0) + f(x_0-c\,t_0)\right] + \frac{1}{2c}\int_{x_0-c\,t_0}^{x_0+c\,t_0} g(s)\,ds.$$

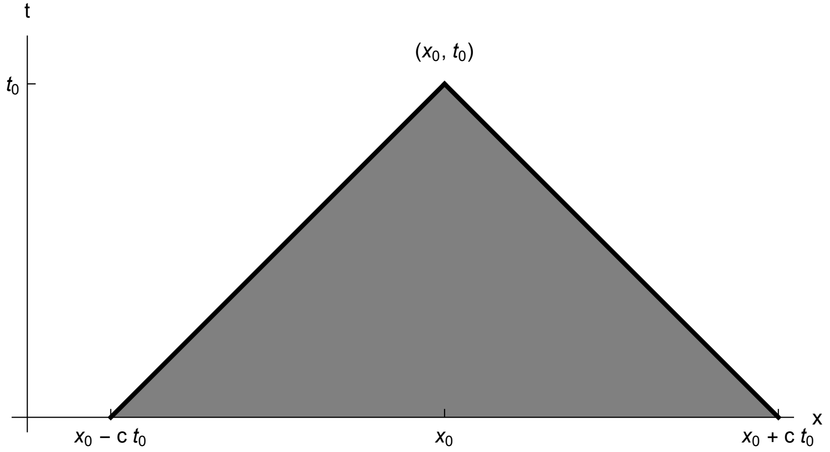
Domain of Dependence (2 of 2)

$$u(x_0, t_0) = \frac{1}{2}\left[f(x_0 + c\,t_0) + f(x_0 - c\,t_0)\right] + \frac{1}{2c}\int_{x_0 - c\,t_0}^{x_0 + c\,t_0} g(s)\,ds.$$

Remarks:

- $u(x_0, t_0)$ depends only on the values of $f(x_0 \pm c\,t)$ and $g(s)$ for $x_0 - c\,t_0 \le s \le x_0 + c\,t_0$.
- The interval $[x_0 - c\,t_0, x_0 + c\,t_0]$ is called the **domain of dependence**.

Domain of Dependence Illustrated

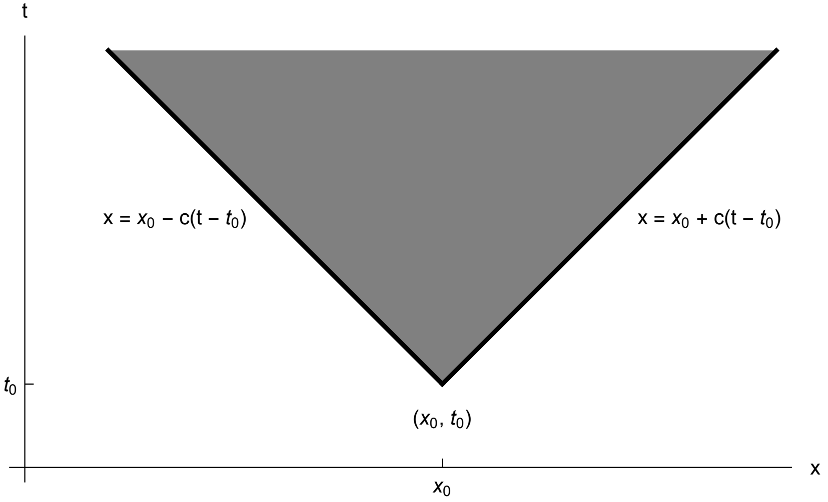


Domain of Influence

The point (x_0, t_0) influences the solution $u(x, t)$ for $t \geq t_0$ at all points between the characteristics passing through (x_0, t_0).

$$\begin{aligned} \frac{t - t_0}{x - x_0} &= \frac{\pm 1}{c} \\ c(t - t_0) &= \pm(x - x_0) \\ \pm x_0 + c(t - t_0) &= \pm x \end{aligned}$$

Domain of Influence Illustrated



Finite Length String

D'Alembert's solution to the wave equation can be adapted to the wave equation with $0 < x < L$.

$$\begin{aligned}
u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\
u(0,t) &= u(L,t) = 0 \\
u(x,0) &= f(x) \\
u_t(x,0) &= g(x)
\end{aligned}$$

Case: Plucked String

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\
u(0,t) &= u(L,t) = 0 \\
u(x,0) &= f(x) \\
u_t(x,0) &= 0
\end{aligned}
$$

We have used separation of variables and Fourier series to determine

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} a_n \cos\frac{c\,n\,\pi\,t}{L} \sin\frac{n\,\pi\,x}{L} \\
&= \frac{1}{2}\left[\sum_{n=1}^{\infty} a_n \sin\frac{n\pi(x+c\,t)}{L} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi(x-c\,t)}{L}\right] \\
&= \frac{1}{2}\left[f(x+c\,t) + f(x-c\,t)\right],
\end{aligned}
$$

where $f(x)$ is the odd, $2L$-periodic extension of the initial displacement.

Case: Struck String

$$\begin{aligned}
u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\
u(0,t) &= u(L,t) = 0 \\
u(L,t) &= 0 \\
u(x,0) &= 0 \\
u_t(x,0) &= g(x)
\end{aligned}$$

We have used separation of variables and Fourier series to determine

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} b_n \sin \frac{c\, n\, \pi\, t}{L} \sin \frac{n\, \pi\, x}{L} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[b_n \cos \frac{n\pi(x - c\, t)}{L} - b_n \cos \frac{n\pi(x + c\, t)}{L} \right].
\end{aligned}$$

Integrating Term by Term

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\sum_{n=1}^{\infty}\left[b_n \cos\frac{n\pi(x-c\,t)}{L} - b_n \cos\frac{n\pi(x+c\,t)}{L}\right] \\
&= \frac{1}{2}\sum_{n=1}^{\infty} b_n \frac{n\pi}{L}\int_{x-c\,t}^{x+c\,t} \sin\frac{n\pi s}{L}\,ds \\
&= \frac{1}{2}\int_{x-c\,t}^{x+c\,t}\left(\sum_{n=1}^{\infty}\left[b_n\frac{n\pi}{L}\right]\sin\frac{n\pi s}{L}\right)\,ds \\
&= \frac{1}{2c}\int_{x-c\,t}^{x+c\,t}\left(\sum_{n=1}^{\infty}\left[b_n\frac{cn\pi}{L}\right]\sin\frac{n\pi s}{L}\right)\,ds \\
&= \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} g(s)\,ds
\end{aligned}
$$

where $g(x)$ is the odd, $2L$-periodic extension of the initial velocity.

Example

Find the solution to the initial boundary value problem

$$\begin{aligned} u_{tt} &= 4u_{xx} \text{ for } 0 < x < 1 \text{ and } t > 0 \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= \begin{cases} 0 & \text{if } x < 1/4 \\ 1 & \text{if } 1/4 \leq x \leq 3/4 \\ 0 & \text{if } 3/4 < x < 1. \end{cases} \end{aligned}$$

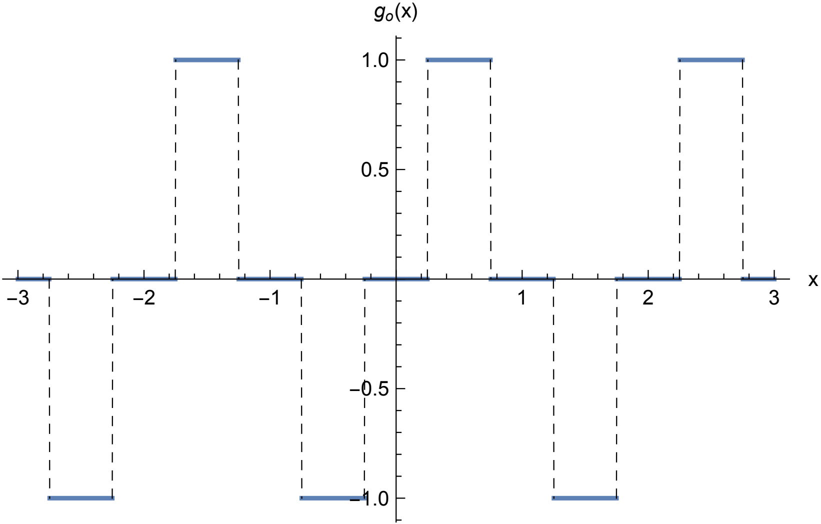
Let $g(x)$ be the odd, 2-periodic extension of $u_t(x,0)$.

Solution (1 of 6)

Let $g_o(x)$ be the odd, 2-periodic extension of $u_t(x,0)$.

$$g_o(x) = \begin{cases} 0 & \text{if } 0 < x < 1/4 \\ 1 & \text{if } 1/4 < x < 3/4 \\ 0 & \text{if } 3/4 < x < 5/4 \\ -1 & \text{if } 5/4 < x < 7/4 \\ 0 & \text{if } 7/4 < x < 2 \end{cases}$$

Solution (2 of 6)



Solution (3 of 6)

Define the function $G(x) = \displaystyle\int_0^x g_o(s)\,ds$.

$$G(x) = \begin{cases} \int_0^x 0\,ds & \text{if } x < 1/4 \\ \int_{1/4}^x 1\,ds & \text{if } 1/4 \le x \le 3/4 \\ \int_{1/4}^{3/4} 1\,ds & \text{if } 3/4 < x < 5/4 \\ \int_{1/4}^{3/4} 1\,ds + \int_{5/4}^x (-1)\,ds & \text{if } 5/4 < x < 7/4 \\ \int_{1/4}^{3/4} 1\,ds + \int_{5/4}^{7/4} (-1)\,ds & \text{if } 7/4 < x < 2 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x < 1/4 \\ x - 1/4 & \text{if } 1/4 \le x \le 3/4 \\ 1/2 & \text{if } 3/4 < x < 5/4 \\ -x + 7/4 & \text{if } 5/4 < x < 7/4 \\ 0 & \text{if } 7/4 < x < 2 \end{cases}$$

Solution (4 of 6)

$$G(x+2t) = \begin{cases} 0 & \text{if } x+2t < 1/4 \\ x+2t-1/4 & \text{if } 1/4 \leq x+2t \leq 3/4 \\ 1/2 & \text{if } 3/4 < x+2t < 5/4 \\ -x-2t+7/4 & \text{if } 5/4 < x+2t < 7/4 \\ 0 & \text{if } 7/4 < x+2t < 2 \end{cases}$$

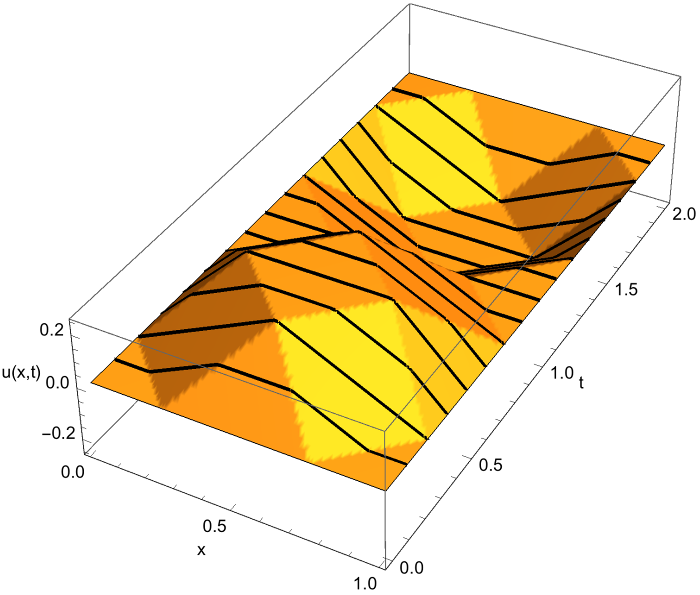
$$G(x-2t) = \begin{cases} 0 & \text{if } x-2t < 1/4 \\ x-2t-1/4 & \text{if } 1/4 \leq x-2t \leq 3/4 \\ 1/2 & \text{if } 3/4 < x-2t < 5/4 \\ -x+2t+7/4 & \text{if } 5/4 < x-2t < 7/4 \\ 0 & \text{if } 7/4 < x-2t < 2 \end{cases}$$

Solution (5 of 6)

Using d'Alembert's solution to the wave equation, then

$$\begin{aligned}
u(x,t) &= \frac{1}{2c}\left[G((x+c\,t) \pmod 2) - G((x-c\,t) \pmod 2)\right] \\
&= \frac{1}{4}\left[G((x+2t) \pmod 2) - G((x-2t) \pmod 2)\right].
\end{aligned}$$

Solution (6 of 6)



Homework

- Read Sections 5.2 and 5.3
- Exercises: 6–10