

D'Alembert's Solution to the Wave Equation

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn:

- ▶ a change of variable technique which simplifies the wave equation,
- ▶ d'Alembert's solution to the wave equation which avoids the summing of a Fourier series solution.

Wave Equation

Consider the initial value problem for the unbounded, homogeneous one-dimensional wave equation

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\u(x, 0) &= f(x) \\u_t(x, 0) &= g(x).\end{aligned}$$

Rewrite the PDE by making the change of variables

$$\begin{aligned}\xi &= x + c t \\ \eta &= x - c t.\end{aligned}$$

Change of Variables

First partial derivatives:

$$\begin{aligned}u_x &= u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta \\ u_t &= u_\xi \xi_t + u_\eta \eta_t = c(u_\xi - u_\eta)\end{aligned}$$

Second partial derivatives:

$$\begin{aligned}u_{xx} &= u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x + u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\ u_{tt} &= c(u_{\xi\xi} \xi_t + u_{\xi\eta} \eta_t) - c(u_{\eta\xi} \xi_t + u_{\eta\eta} \eta_t) = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})\end{aligned}$$

Substitution into Wave Equation

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \\c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) &= c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \\u_{\xi\eta} &= 0\end{aligned}$$

Integrate both sides of the equation.

$$\begin{aligned}u_{\xi} &= \phi(\xi) \\u(\xi, \eta) &= \int \phi(\xi) d\xi \\&= \Phi(\xi) + \Psi(\eta)\end{aligned}$$

Functions Φ and Ψ are arbitrary smooth functions.

Return to the Original Variables

$$\begin{aligned}u(\xi, \eta) &= \Phi(\xi) + \Psi(\eta) \\u(x, t) &= \Phi(x + ct) + \Psi(x - ct)\end{aligned}$$

This is referred to as **d'Alembert's general solution to the wave equation**.

Question: can Φ and Ψ be chosen to satisfy the initial conditions?

Plucked String (1 of 2)

$$u(x, t) = \Phi(x + ct) + \Psi(x - ct)$$

$$u(x, 0) = \Phi(x) + \Psi(x) = f(x)$$

$$u_t(x, 0) = c\Phi'(x) - c\Psi'(x) = 0$$

$$0 = \Phi'(x) - \Psi'(x)$$

Integrate the last equation.

$$\Phi(x) = \Psi(x) + K$$

where K is an arbitrary constant.

Substituting into the equation for the initial displacement produces

$$2\Psi(x) + K = f(x)$$

$$\Psi(x) = \frac{1}{2} (f(x) - K)$$

$$\Phi(x) = \frac{1}{2} (f(x) + K).$$

Plucked String (2 of 2)

Consequently if f is twice differentiable, then

$$u(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)]$$

solves the initial value problem describing the plucked string.

Struck String (1 of 2)

$$u(x, t) = \Phi(x + ct) + \Psi(x - ct)$$

$$u(x, 0) = \Phi(x) + \Psi(x) = 0$$

$$u_t(x, 0) = c\Phi'(x) - c\Psi'(x) = g(x)$$

Differentiating the 2nd equation reveals $\Phi'(x) = -\Psi'(x)$

Substituting into the 3rd equation produces

$$2c\Phi'(x) = g(x)$$

$$\Phi(x) = \frac{1}{2c} \int_0^x g(s) ds + K$$

$$\Psi(x) = -\frac{1}{2c} \int_0^x g(s) ds - K.$$

Struck String (2 of 2)

Consequently if g is continuously differentiable, then

$$\begin{aligned} u(x, t) &= \frac{1}{2c} \left[\int_0^{x+ct} g(s) ds - \int_0^{x-ct} g(s) ds \right] \\ &= \frac{1}{2c} \left[\int_0^{x+ct} g(s) ds + \int_{x-ct}^0 g(s) ds \right] \\ &= \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds \end{aligned}$$

solves the initial value problem describing the struck string.

Nonzero Displacement and Velocity

By the Principle of Superposition, the general solution is

$$u(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

Example: Plucked String

Determine the solution to the initial value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ u(x, 0) &= f(x) = \begin{cases} 2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \\ u_t(x, 0) &= 0 \end{aligned}$$

Solution (1 of 7)

Using d'Alembert's solution

$$u(x, t) = \frac{1}{2} [f(x + c t) + f(x - c t)].$$

Note:

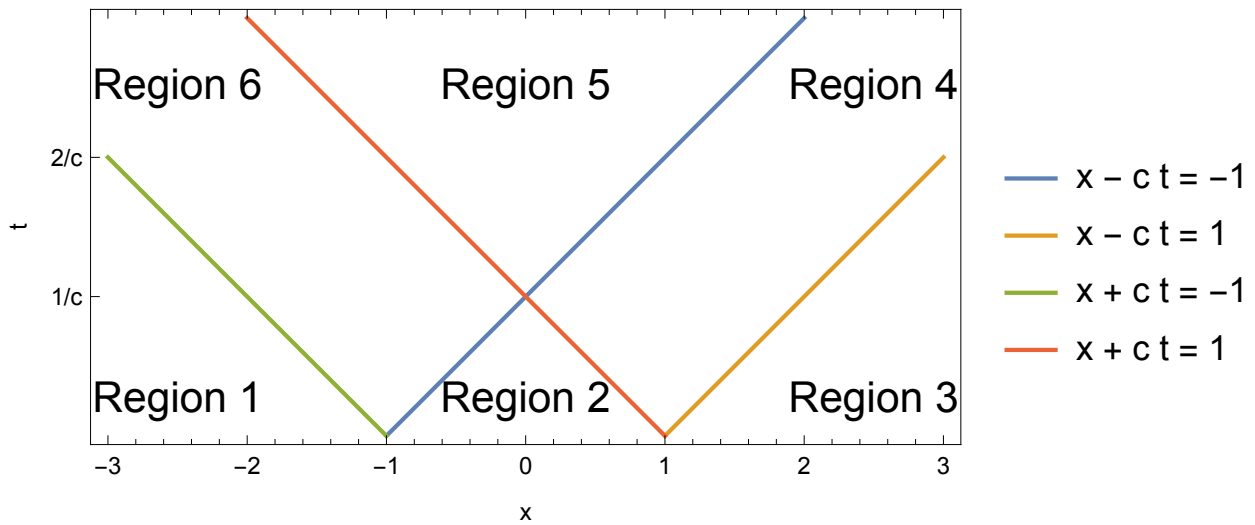
- ▶ Along lines where $x + c t$ is constant the term $f(x + c t)$ is constant.
- ▶ Likewise along lines where $x - c t$ is constant the term $f(x - c t)$ is constant.
- ▶ These lines are called **characteristics**.

Solution (2 of 7)

$$\begin{aligned} f(x) &= \begin{cases} 2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \\ f(x + c t) &= \begin{cases} 2 & \text{if } -1 < x + c t < 1 \\ 0 & \text{otherwise} \end{cases} \\ \frac{1}{2} f(x + c t) &= \begin{cases} 1 & \text{if } -1 - c t < x < 1 - c t \\ 0 & \text{otherwise} \end{cases} \\ \frac{1}{2} f(x - c t) &= \begin{cases} 1 & \text{if } -1 + c t < x < 1 + c t \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Remark: the characteristics where $x + c t = \pm 1$ and $x - c t = \pm 1$ help determine the solution.

Solution (3 of 7)



Solution (4 of 7)

Region 1: $\{(x, t) \mid x + ct < -1\}$

Region 2: $\{(x, t) \mid -1 < x - ct \text{ and } x + ct < 1\}$

Region 3: $\{(x, t) \mid 1 < x - ct\}$

Region 4: $\{(x, t) \mid 1 < x + ct \text{ and } -1 < x - ct < 1\}$

Region 5: $\{(x, t) \mid 1 < x + ct \text{ and } x - ct < -1\}$

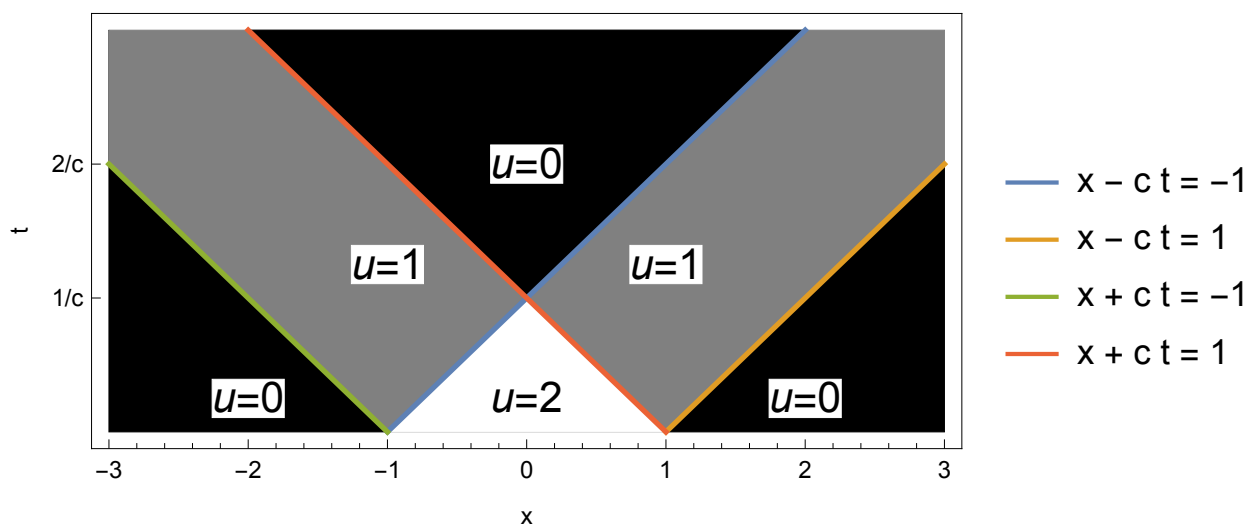
Region 6: $\{(x, t) \mid -1 < x + ct < 1 \text{ and } x - ct < -1\}$

Solution (5 of 7)

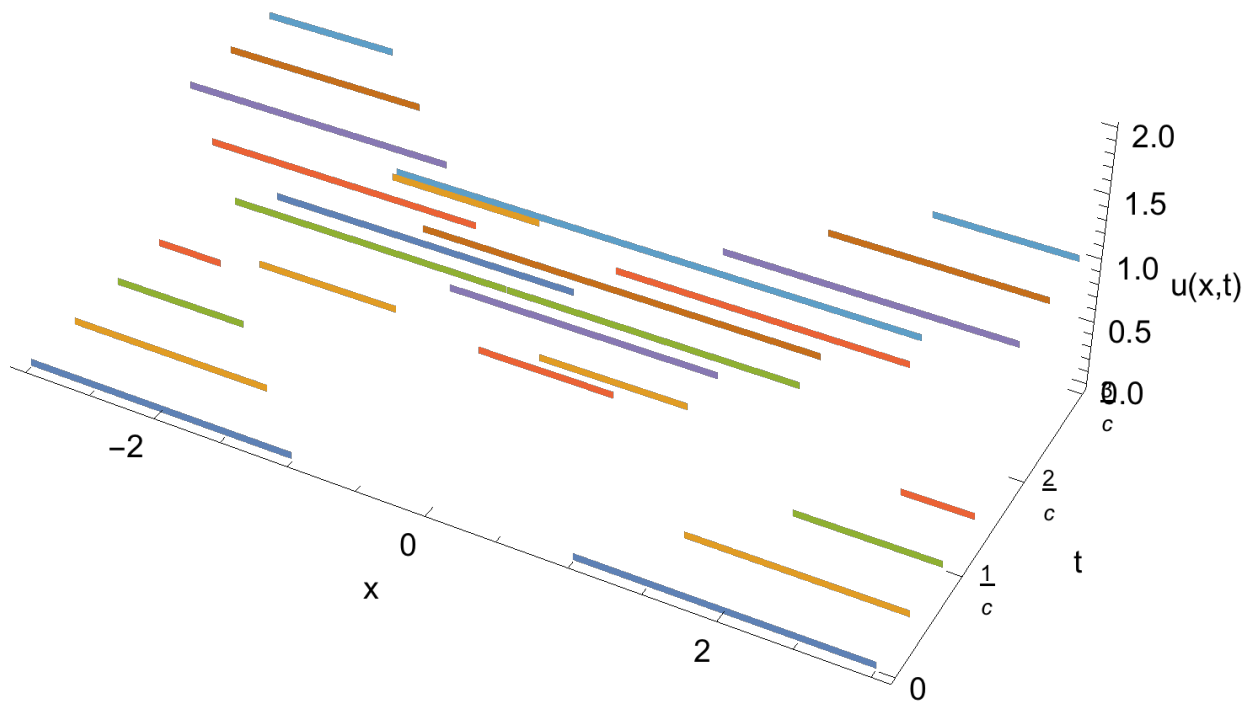
$$u(x, t) = \frac{1}{2}f(x + ct) + \frac{1}{2}f(x - ct)$$

$$= \begin{cases} 0 & \text{if } x + ct < -1 \\ 2 & \text{if } -1 < x - ct < 1 \text{ and } -1 < x + ct < 1 \\ 0 & \text{if } 1 < x - ct \\ 1 & \text{if } 1 < x + ct \text{ and } -1 < x - ct < 1 \\ 0 & \text{if } 1 < x + ct \text{ and } x - ct < -1 \\ 1 & \text{if } -1 < x + ct < 1 \text{ and } x - ct < -1 \end{cases}$$

Solution (6 of 7)



Solution (7 of 7)



Example: Struck String

Determine the solution to the initial value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ u(x, 0) &= 0 \\ u_t(x, 0) &= g(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Solution (1 of 6)

Define the function

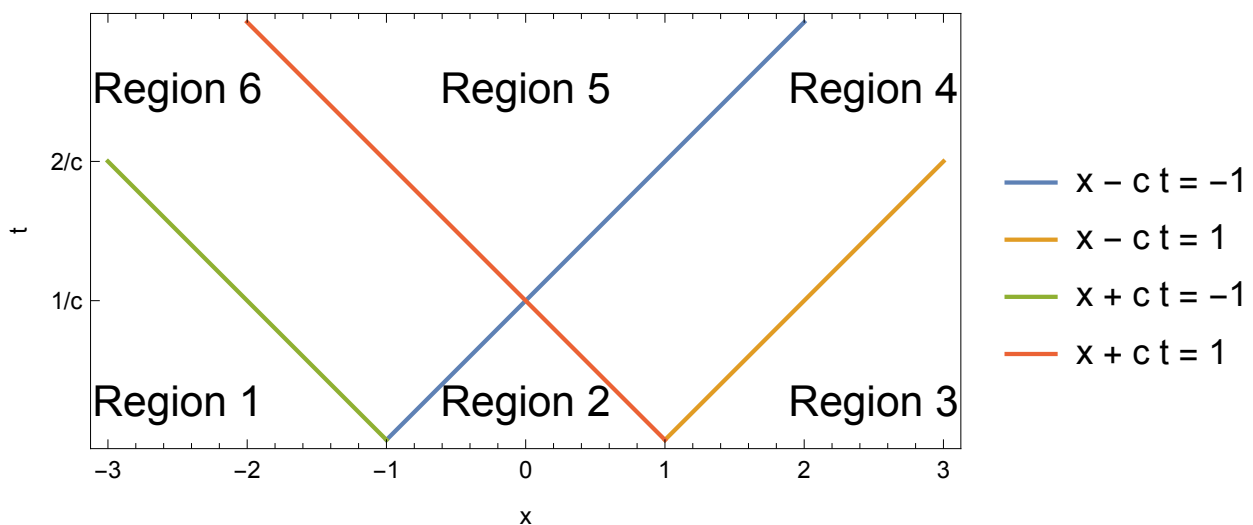
$$G(z) = \int_0^z g(w) dw = \begin{cases} -1 & \text{if } z < -1 \\ z & \text{if } -1 \leq z \leq 1 \\ 1 & \text{if } z > 1. \end{cases}$$

then

$$u(x, t) = \frac{1}{2c} [G(x + ct) - G(x - ct)].$$

As in the previous example, the characteristics $x + ct = \pm 1$ and $x - ct = \pm 1$ divide the xt -plane into six regions.

Solution (2 of 6)



Solution (3 of 6)

Region 1: $\{(x, t) \mid x + ct < -1\}$

Region 2: $\{(x, t) \mid -1 < x - ct \text{ and } x + ct < 1\}$

Region 3: $\{(x, t) \mid 1 < x - ct\}$

Region 4: $\{(x, t) \mid 1 < x + ct \text{ and } -1 < x - ct < 1\}$

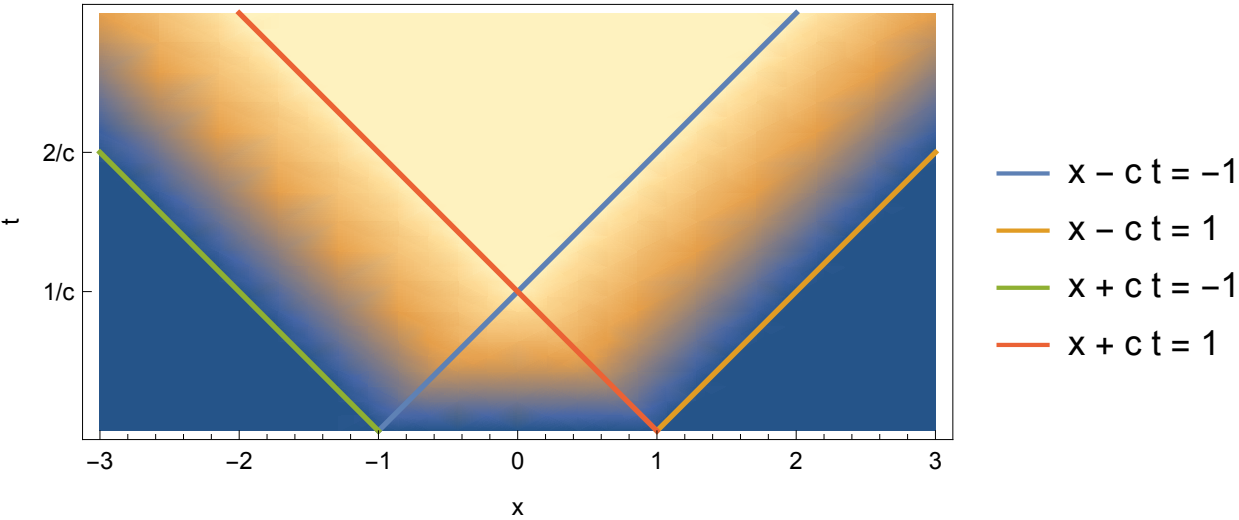
Region 5: $\{(x, t) \mid 1 < x + ct \text{ and } x - ct < -1\}$

Region 6: $\{(x, t) \mid -1 < x + ct < 1 \text{ and } x - ct < -1\}$

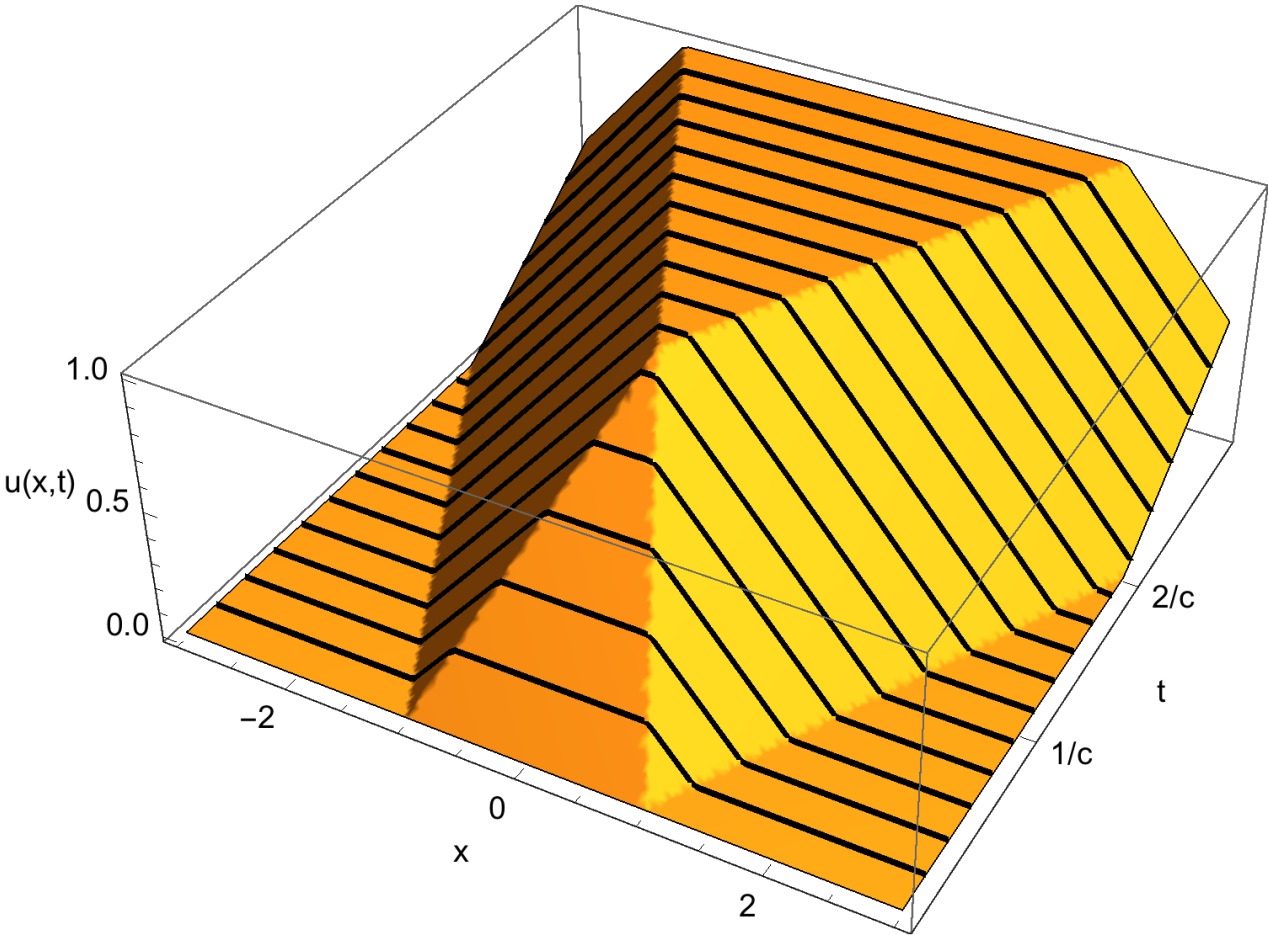
Solution (4 of 6)

$$\begin{aligned} u(x, t) &= \frac{1}{2c} G(x + ct) - \frac{1}{2c} G(x - ct) \\ &= \frac{1}{2c} \begin{cases} 0 & \text{if } x + ct < -1 \\ 2ct & \text{if } -1 < x - ct \text{ and } x + ct < 1 \\ 0 & \text{if } 1 < x - ct \\ 1 - x + ct & \text{if } 1 < x + ct \text{ and } -1 < x - ct < 1 \\ 2 & \text{if } 1 < x + ct \text{ and } x - ct < -1 \\ 1 + x + ct & \text{if } x - ct < -1 \text{ and } -1 < x + ct < 1 \end{cases} \end{aligned}$$

Solution (5 of 6)



Solution (6 of 6)



Domain of Dependence (1 of 2)

In general the solution to the initial value problem:

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\u(x, 0) &= f(x) \\u_t(x, 0) &= g(x)\end{aligned}$$

can be expressed as

$$u(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

At the point (x_0, t_0) then

$$u(x_0, t_0) = \frac{1}{2} [f(x_0 + ct_0) + f(x_0 - ct_0)] + \frac{1}{2c} \int_{x_0-ct_0}^{x_0+ct_0} g(s) ds.$$

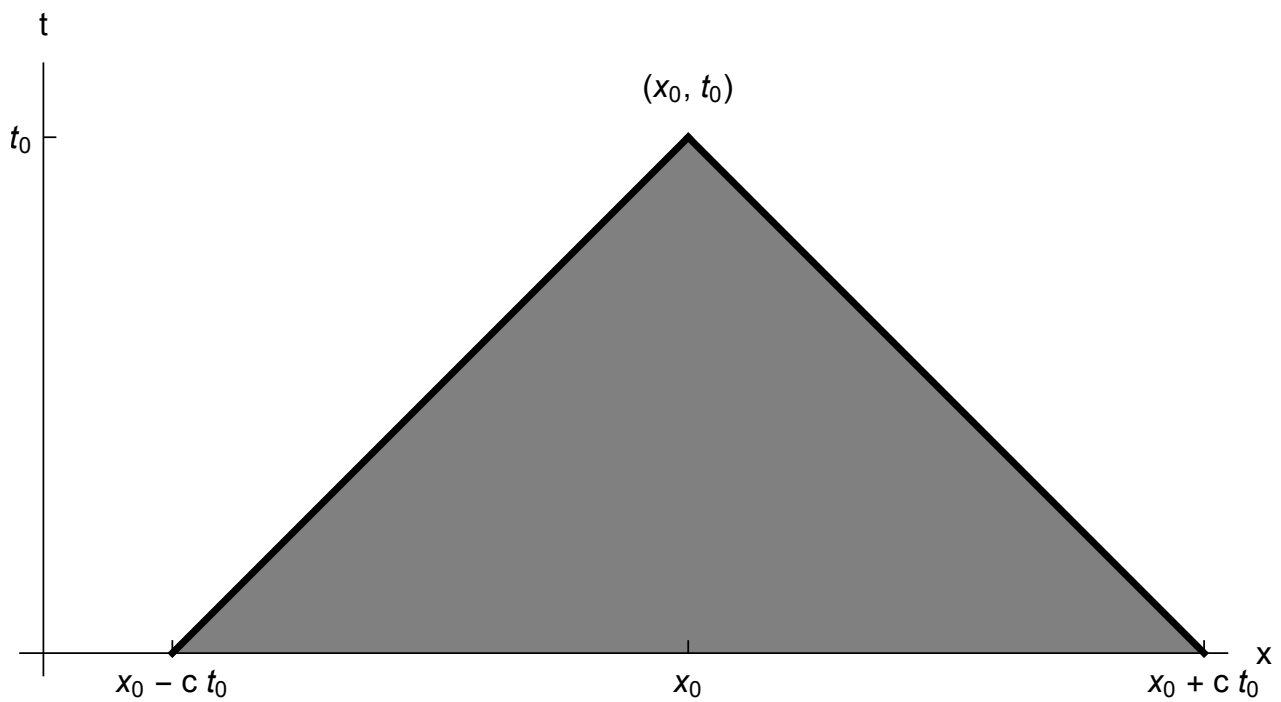
Domain of Dependence (2 of 2)

$$u(x_0, t_0) = \frac{1}{2} [f(x_0 + ct_0) + f(x_0 - ct_0)] + \frac{1}{2c} \int_{x_0-ct_0}^{x_0+ct_0} g(s) ds.$$

Remarks:

- ▶ $u(x_0, t_0)$ depends only on the values of $f(x_0 \pm ct)$ and $g(s)$ for $x_0 - ct_0 \leq s \leq x_0 + ct_0$.
- ▶ The interval $[x_0 - ct_0, x_0 + ct_0]$ is called the **domain of dependence**.

Domain of Dependence Illustrated



Domain of Influence

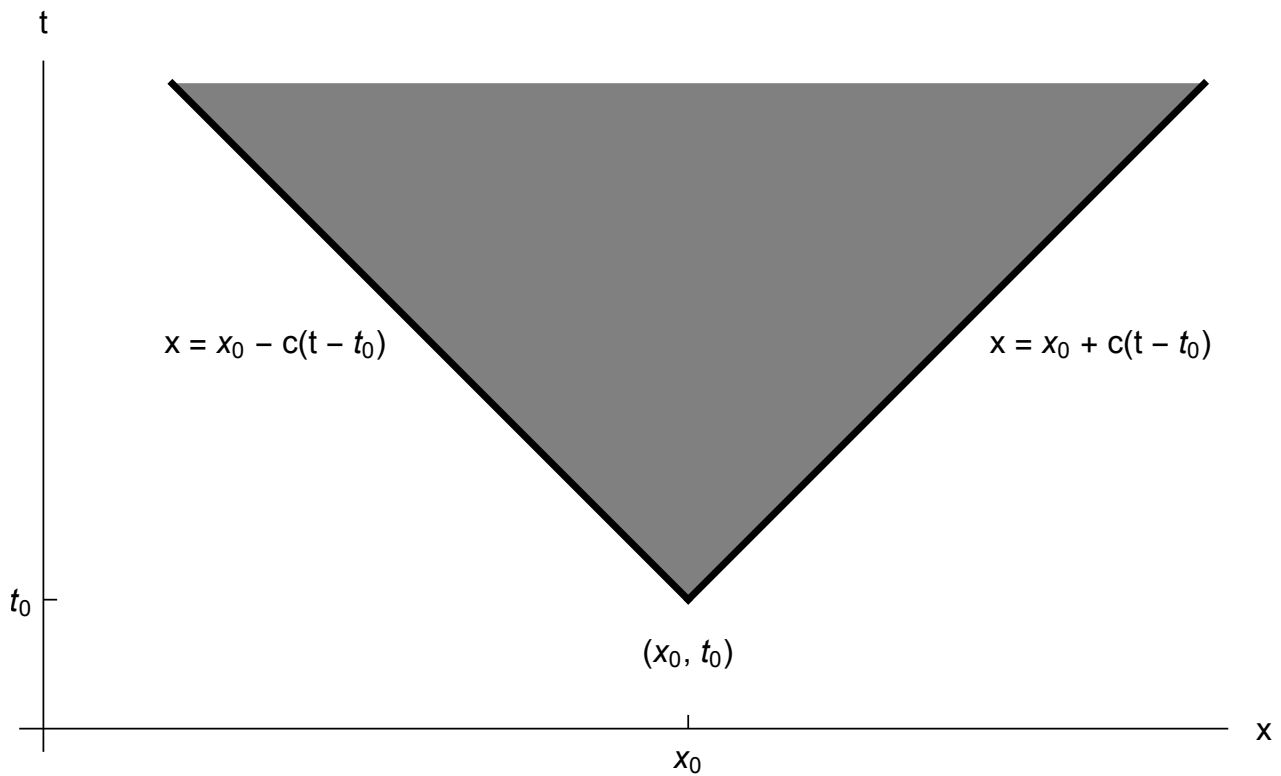
The point (x_0, t_0) influences the solution $u(x, t)$ for $t \geq t_0$ at all points between the characteristics passing through (x_0, t_0) .

$$\frac{t - t_0}{x - x_0} = \frac{\pm 1}{c}$$

$$c(t - t_0) = \pm(x - x_0)$$

$$\pm x_0 + c(t - t_0) = \pm x$$

Domain of Influence Illustrated



Finite Length String

D'Alembert's solution to the wave equation can be adapted to the wave equation with $0 < x < L$.

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\u(0, t) &= u(L, t) = 0 \\u(x, 0) &= f(x) \\u_t(x, 0) &= g(x)\end{aligned}$$

Case: Plucked String

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\u(0, t) &= u(L, t) = 0 \\u(x, 0) &= f(x) \\u_t(x, 0) &= 0\end{aligned}$$

We have used separation of variables and Fourier series to determine

$$\begin{aligned}u(x, t) &= \sum_{n=1}^{\infty} a_n \cos \frac{c n \pi t}{L} \sin \frac{n \pi x}{L} \\&= \frac{1}{2} \left[\sum_{n=1}^{\infty} a_n \sin \frac{n \pi (x + c t)}{L} + \sum_{n=1}^{\infty} a_n \sin \frac{n \pi (x - c t)}{L} \right] \\&= \frac{1}{2} [f(x + c t) + f(x - c t)],\end{aligned}$$

where $f(x)$ is the odd, $2L$ -periodic extension of the initial displacement.

Case: Struck String

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\u(0, t) &= u(L, t) = 0 \\u(L, t) &= 0 \\u(x, 0) &= 0 \\u_t(x, 0) &= g(x)\end{aligned}$$

We have used separation of variables and Fourier series to determine

$$\begin{aligned}u(x, t) &= \sum_{n=1}^{\infty} b_n \sin \frac{c n \pi t}{L} \sin \frac{n \pi x}{L} \\&= \frac{1}{2} \sum_{n=1}^{\infty} \left[b_n \cos \frac{n \pi (x - c t)}{L} - b_n \cos \frac{n \pi (x + c t)}{L} \right].\end{aligned}$$

Integrating Term by Term

$$\begin{aligned}u(x, t) &= \frac{1}{2} \sum_{n=1}^{\infty} \left[b_n \cos \frac{n\pi(x - ct)}{L} - b_n \cos \frac{n\pi(x + ct)}{L} \right] \\&= \frac{1}{2} \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \int_{x-ct}^{x+ct} \sin \frac{n\pi s}{L} ds \\&= \frac{1}{2} \int_{x-ct}^{x+ct} \left(\sum_{n=1}^{\infty} \left[b_n \frac{n\pi}{L} \right] \sin \frac{n\pi s}{L} \right) ds \\&= \frac{1}{2c} \int_{x-ct}^{x+ct} \left(\sum_{n=1}^{\infty} \left[b_n \frac{cn\pi}{L} \right] \sin \frac{n\pi s}{L} \right) ds \\&= \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds\end{aligned}$$

where $g(x)$ is the odd, $2L$ -periodic extension of the initial velocity.

Example

Find the solution to the initial boundary value problem

$$\begin{aligned}u_{tt} &= 4u_{xx} \text{ for } 0 < x < 1 \text{ and } t > 0 \\u(0, t) &= u(L, t) = 0 \\u(x, 0) &= 0 \\u_t(x, 0) &= \begin{cases} 0 & \text{if } x < 1/4 \\ 1 & \text{if } 1/4 \leq x \leq 3/4 \\ 0 & \text{if } 3/4 < x < 1. \end{cases}\end{aligned}$$

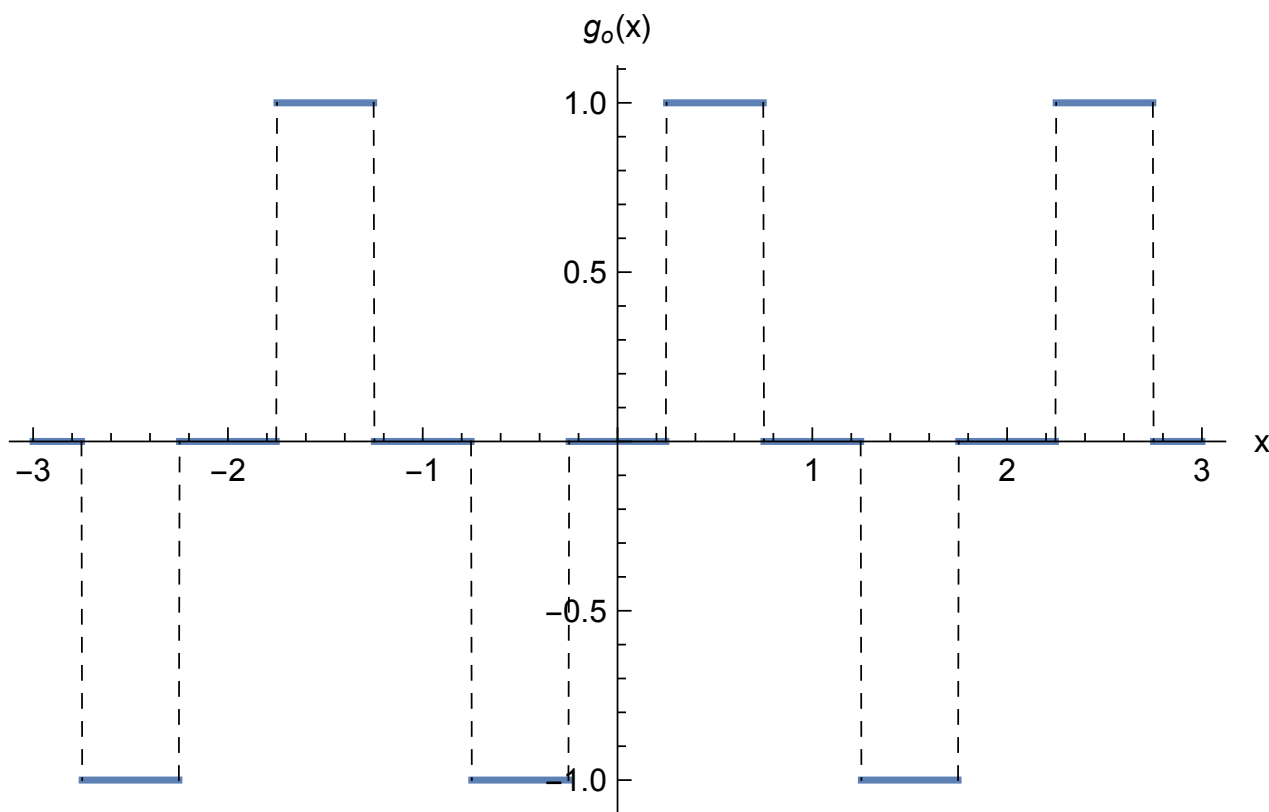
Let $g(x)$ be the odd, 2-periodic extension of $u_t(x, 0)$.

Solution (1 of 6)

Let $g_o(x)$ be the odd, 2-periodic extension of $u_t(x, 0)$.

$$g_o(x) = \begin{cases} 0 & \text{if } 0 < x < 1/4 \\ 1 & \text{if } 1/4 < x < 3/4 \\ 0 & \text{if } 3/4 < x < 5/4 \\ -1 & \text{if } 5/4 < x < 7/4 \\ 0 & \text{if } 7/4 < x < 2 \end{cases}$$

Solution (2 of 6)



Solution (3 of 6)

Define the function $G(x) = \int_0^x g_0(s) ds$.

$$G(x) = \begin{cases} \int_0^x 0 ds & \text{if } x < 1/4 \\ \int_{1/4}^x 1 ds & \text{if } 1/4 \leq x \leq 3/4 \\ \int_{1/4}^{3/4} 1 ds & \text{if } 3/4 < x < 5/4 \\ \int_{1/4}^{3/4} 1 ds + \int_{5/4}^x (-1) ds & \text{if } 5/4 < x < 7/4 \\ \int_{1/4}^{3/4} 1 ds + \int_{5/4}^{7/4} (-1) ds & \text{if } 7/4 < x < 2 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x < 1/4 \\ x - 1/4 & \text{if } 1/4 \leq x \leq 3/4 \\ 1/2 & \text{if } 3/4 < x < 5/4 \\ -x + 7/4 & \text{if } 5/4 < x < 7/4 \\ 0 & \text{if } 7/4 < x < 2 \end{cases}$$

Solution (4 of 6)

$$G(x + 2t) = \begin{cases} 0 & \text{if } x + 2t < 1/4 \\ x + 2t - 1/4 & \text{if } 1/4 \leq x + 2t \leq 3/4 \\ 1/2 & \text{if } 3/4 < x + 2t < 5/4 \\ -x - 2t + 7/4 & \text{if } 5/4 < x + 2t < 7/4 \\ 0 & \text{if } 7/4 < x + 2t < 2 \end{cases}$$

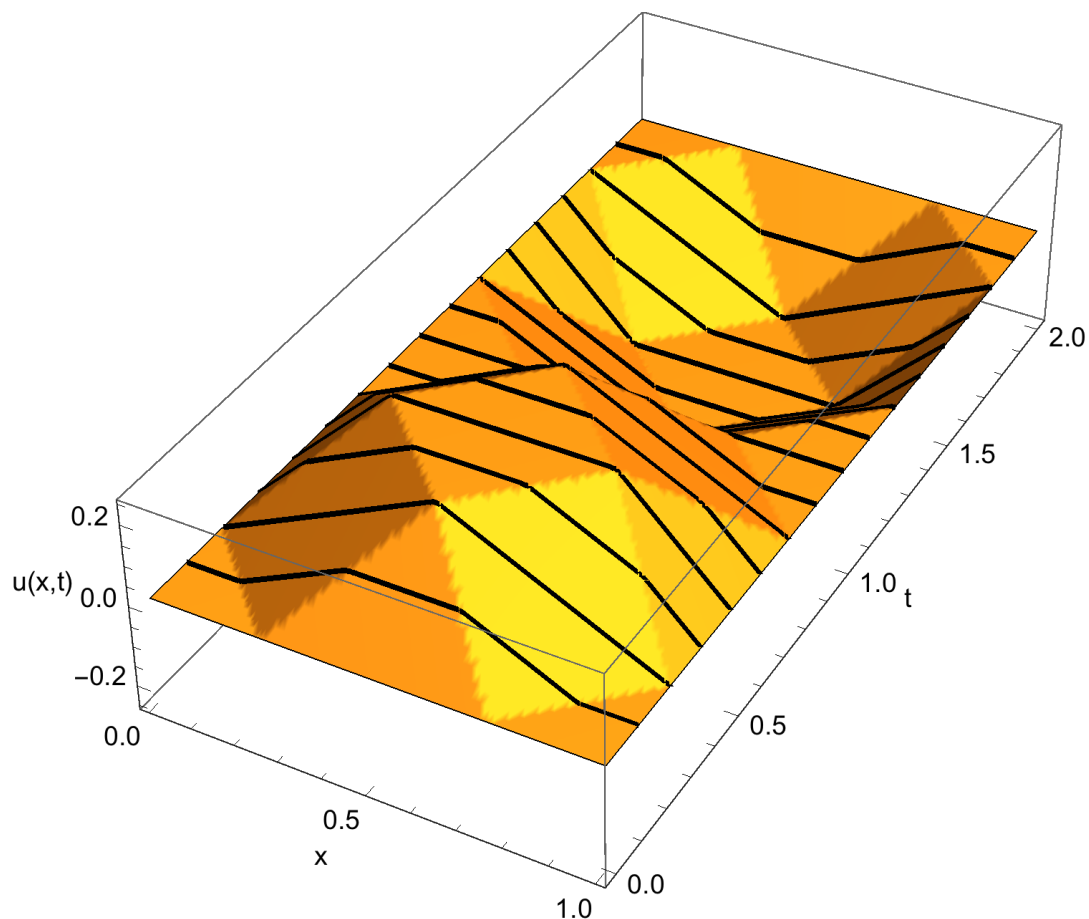
$$G(x - 2t) = \begin{cases} 0 & \text{if } x - 2t < 1/4 \\ x - 2t - 1/4 & \text{if } 1/4 \leq x - 2t \leq 3/4 \\ 1/2 & \text{if } 3/4 < x - 2t < 5/4 \\ -x + 2t + 7/4 & \text{if } 5/4 < x - 2t < 7/4 \\ 0 & \text{if } 7/4 < x - 2t < 2 \end{cases}$$

Solution (5 of 6)

Using d'Alembert's solution to the wave equation, then

$$\begin{aligned} u(x, t) &= \frac{1}{2c} [G((x + ct) \pmod{2}) - G((x - ct) \pmod{2})] \\ &= \frac{1}{4} [G((x + 2t) \pmod{2}) - G((x - 2t) \pmod{2})]. \end{aligned}$$

Solution (6 of 6)



Homework

- ▶ Read Sections 5.2 and 5.3
- ▶ Exercises: 6–10